

Fundamental Theorems for Capitalist Reproduction Systems with Heterogeneous Labour

Takao Fujimoto and Ravindra R. Ranade*

Abstract This paper proposes to take a new look at the Fundamental Marxian Theorems, and establish some propositions for capitalist reproduction systems in a linear input-output model with heterogeneous labour. In so doing, we make a slightly stronger assumption concerning the productiveness of a given system, and obtain a theorem that each type of labour gets exploited in the system, if the wage rate of that labour type is not high enough to make savings, irrespective of the levels of profit rates of production processes. Some more related propositions are also presented.

1 Introduction

This paper takes a new look at the Fundamental Marxian Theorems (FMT's) based upon the method and the results in Fujimoto and Opocher ([14]). The FMT so named by Morishima ([24]) is concerned with the relationships between the profit rates and the existence of exploitation. Notable contributions for models with heterogeneous labour are Potron ([31]), Okishio ([28]), Morishima ([24], [26]), Bowles and Gintis ([1], [2], [3]), Steedman ([32], [33]),

*Faculty of Economics, Fukuoka University, Fukuoka, Japan, and Faculty of Economics, Kagawa University, Japan

Hollander ([15]), Krause ([16], [17], [18]), and Fujimori ([7]). Most importantly, the FMT's assert that positive profits in *every* process imply exploitation in at least one type of labour, or when reduction to 'abstract labour' is performed, the positivity of the equilibrium rate of profit is equivalent to the existence of exploitation of abstract labour.¹

If the FMT is valid only in equilibrium, it cannot claim a wider applicability to the real world. One of the present authors established an FMT in disequilibrium for models with joint production and homogeneous labour (Fujimoto ([8])). A theorem in Fujimoto ([8]) states that if the *social average* rate of profit is positive, labour is exploited, thus allowing for negative profit rates in some industries.² Another theorem therein asserts, even in a model with joint production, that if the *social average* rate of growth is positive, labour is exploited. Besides, we have a question: in a recession when many industries suffer from losses, is there no exploitation? FMT's cannot give a straight answer to this natural question. Washida ([37]) was a great step forward, and yet things remain somewhat blurred.

On the other hand, as Bowles and Gintis ([1]) argue, we should take into consideration the heterogeneity of labour for a model to reflect the ever-growing division of labour in the modern economies.

In this article, using a model with heterogeneous labour, we wish to establish some propositions which show the existence of exploitation for each type of labour under a productiveness assumption made slightly stronger than those in the literature, and these propositions hold whether profit rates are positive or not. Our results in mathematical terms are in close parallel to those obtained for models with homogeneous labour under the name of the Generalized

¹Exceptionally, the Generalized Fundamental Marxian Theorem by Washida ([37]) and Matsuo ([21]) can deal with disequilibrium states with negative profit rates. It seems, however, that their method may not be carried over to models with heterogenous labour in a simple way.

²See footnote 1 above.

Fundamental Marxian Theorem, e.g., by Washida ([37]) and Matsuo ([21]). In section 2, our model is explained, and in the following section 3, some propositions are presented. Section 4 gives a concrete numerical example based on Bowles and Gintis ([1]). The important point to note is that no reduction to abstract labour is necessary, nor required the actual sizes of employment of various types of labour. The final section includes several remarks.

2 Labour Values for a Model with Heterogeneous Labour

Let us consider a simple Leontief-type input-output model without joint production and with heterogeneous labour. There are n kinds of normal commodities, and m types of labour services.³ A worker of type i renders labour service of type i . We adopt all the symbols in Krause ([18]) except one. They are:

- A : a given $n \times n$ matrix of material input coefficients,
 processes as columns,
- L : a given $m \times n$ matrix labour input coefficients,
- C : a given $n \times m$ matrix of consumptions per unit
 of labour service produced
- I_n : the $n \times n$ identity matrix,
- Λ : the $m \times n$ matrix, $L(I - A)^{-1}$.

For comparison between vectors, $x, y \in R^n$, $x \ll y$ means $x_j < y_j$ for all j ; $x < y$ means $x_j \leq y_j$ for all j and $x \neq y$; $x \leq y$ means $x_j \leq y_j$ for all j .

³In what follows, ‘a commodity’ stand for a normal commodity or a labour service, while the phrase ‘a normal commodity’ excludes the latter.

In a way similar to models with homogeneous labour, an important role was played by the matrices, $\Lambda \equiv L(I - A)^{-1}$ and $H \equiv L(I - A)^{-1}C$. Notable authors, however, have given an incorrect interpretation of Λ or H .

Bowles and Gintis([1], p.186) wrote: “Let the value of good i be given by $\lambda_i = (\lambda_{1i}, \dots, \lambda_{mi})$, a vector of the direct and indirect labour hours of each of our m types embodied in a unit of good i . If we let $\Lambda = (\lambda_{ri})$, then we have $\Lambda = \Lambda A + L$, so $\Lambda = L(I - A)^{-1}$. We call λ_{ri} the ‘ r -value of good i ’.”

Krause([18], p.174) also stated: “the value/surplus-aspect can be analyzed in terms of the $(m \times n)$ -matrix $H = L(I - A)^{-1}C$ as follows. The entry h_{ij} of H gives the amount of labour of type i required directly or indirectly to reproduce one unit labour-power of type j .”

Potron ([31], p.70) as translated by Mori([22], p.523) put: “[(i, k) -element of $(I - A)^{-1}C$ is the production of [good i] [directly and indirectly] necessary for the subsistence of a consumer of type $[k]$; [(k, k) -element of $L(I - A)^{-1}C$] represents the labour that this subsistence demands [directly and indirectly] from the category of labourers belonging to type $[k]$.”

It is quite clear that each entry of the matrix Λ does not show the labour hours of a particular type required directly and indirectly to produce one unit of a particular commodity. This is simply because in order to produce a commodity, other types of labour may also be necessary, and thus to sustain those types of labour some more commodities, and so the labour type concerned, are required. We should either incorporate the necessary commodities required by other types of labour as material inputs in a way von Neumann adopted ([36]), or devise out a new method as is done in Fujimoto and Opocher ([14]). Let us explain the latter.

We first define some new symbols:

$$\mathbb{I} \equiv \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} \quad \text{and} \quad \mathbb{A} \equiv \begin{pmatrix} A & C \\ L & 0 \end{pmatrix}.$$

We call \mathbb{A} the complete input matrix following Bródy ([5], p.23). In this manner, normal commodities and various labour types are treated in a symmetrical way: the matrix C is the material input coefficient matrix in the household activities to produce labour services. This matrix C is formed by the averages of what have been observed just as A and L are created. When a deviation is excessive in a type of labour, divide this type into two or more categories. Two more symbols are defined with $i > n$.

$$\begin{aligned} \Lambda^{[i]} &\equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \dots, \lambda_{i-1}^{[i]}, \lambda_i^{[i]}, \lambda_{i+1}^{[i]}, \dots, \lambda_{n+m}^{[i]}), \text{ and} \\ \Lambda_{[i]}^{[i]} &\equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \dots, \lambda_{i-1}^{[i]}, 1, \lambda_{i+1}^{[i]}, \dots, \lambda_{n+m}^{[i]}). \end{aligned}$$

The new symbol $\Lambda^{[i]}$ stands for the $(n + m)$ -vector of values in terms of the labour type i . (Following Bowles and Gintis, we call $\lambda_j^{[i]}$ the i -value of commodity j .) Then, this should satisfy

$$\Lambda^{[i]} = \Lambda_{[i]}^{[i]} \cdot \mathbb{A}. \tag{1}$$

This equation simply says that the value of a commodity or a type of labour is the labour hours of type i required directly or indirectly to produce one unit of that commodity.⁴

Now the problem is how to solve, in a systematic way, eq.(1) of $(n + m)$ -equations with the same number of unknowns. Eq.(1) can be transformed to

$$\Lambda_{[i]}^{[i]} = \Lambda_{[i]}^{[i]} \cdot \mathbb{A} + (0, 0, \dots, 0, 1 - \lambda_i^{[i]}, 0, \dots, 0),$$

which is next rewritten as follows:

$$v \cdot \Lambda_{[i]}^{[i]} = v \cdot \Lambda_{[i]}^{[i]} \cdot \mathbb{A} + e_i, \tag{2}$$

⁴In a simple Leontief model with homogeneous labour, eq.(1) corresponds to the well-known system of equations, $\Lambda = \Lambda A + L$ and $\lambda_\ell = \Lambda C$, where Λ is the row-vector of labour values of ordinary commodities and λ_ℓ is the labour value of homogeneous labour.

where v is a positive scalar and e_i is the row $(n+m)$ -vector whose i -th entry is unity with the remaining elements being all zero. Note that v has to satisfy

$$v \cdot (1 - \lambda_i^{[i]}) = 1, \tag{3}$$

and can be positive only when $1 > \lambda_i^{[i]} \geq 0$, which in turn leads to $v \geq 1$. Define

$$q \equiv v \cdot \Lambda_{[i]}^{[i]},$$

and we have from eq.(2),

$$q = q \cdot \mathbb{A} + e_i. \tag{4}$$

Therefore,

$$q \equiv e_i \cdot (\mathbb{I} - \mathbb{A})^{-1}. \tag{5}$$

From eqs.(3) and (5), we can solve $\Lambda^{[i]}$ as

$$\lambda_i^{[i]} = \frac{q_i - 1}{q_i} \quad \text{and} \quad \lambda_j^{[i]} = \frac{q_j}{q_i}. \tag{6}$$

From this eq.(5), it follows that $q_i \geq 1$, and so we have $1 > \lambda_i^{[i]} \geq 0$. Thus, we can retrace our steps back to the original eq.(1), showing the legitimacy of our solution.

In sum, to have meaningful values, all we have to assume is **Productiveness Assumption**: There exists an $x \in R_+^{n+m}$ such that

$$x \gg \mathbb{A}x.$$

This assumption is certainly stronger than the usual one that there exists an $x \in R_+^n$ such that

$$x \gg Ax,$$

and the reader may be tempted to think that our assumption has already implied the existence of exploitation.⁵ We, however, simply treat normal commodities and labour in a symmetric way, and extend the reproducibility also to all types of labour. Therefore, it is not a tautology to prove the existence of exploitation from our assumption.

3 Propositions and Theorems

From what has been explained in the previous section, we have our first proposition.

Proposition 1. Given the Productiveness Assumption, the labour value of labour of each type in terms of itself is nonnegative and less than unity.

We denote by w_i the wage rate per one unit of labour type i , and suppose that in our economy a row n -vector of prices for normal commodities, $p \in R_+^n$, is prevailing. Let \mathbb{A}_j , C_j , and L_j represent the j -th column of the matrix \mathbb{A} , the j -th column of C , and the j -th row of L , respectively. One more symbol $(p, w) \in R_+^{n+m}$ is defined by juxtaposing two row vectors, $p \in R_+^n$ and $w \in R_+^m$.

To make clear the meaning of the inequality $\lambda_i^{[i]} < 1$, we next present

Proposition 2. The i -value of labour service i means the minimum amount of labour service i necessary to produce one basket C_{i-n} as net output, while reproducing the whole economic system.

Proof. It is well known that q_i in eq.(4) can be obtained by solving the following linear programming problem:

$$(LP) \max q_i \text{ subject to } q \in R_+^{n+m} \text{ and } q \leq q\mathbb{A} + e_i.$$

⁵In our simple model without joint production, it is not difficult to see that if the Frobenius root of the matrix $A + CL$ is smaller than unity, the Productiveness Assumption here is satisfied.

Dual to this is the problem:

$$(DP) \quad \min x_i \quad \text{subject to } x \in R_+^{n+m} \quad \text{and } x \geq \mathbb{A}x + e_i.$$

Now the minimum amount of labour service i necessary to produce one basket C_{i-n} as net output, while reproducing the whole economic system, can be calculated by solving the problem (VP):

$$(VP) \quad \min L_{i-n}y \quad \text{subject to } y \in R_+^n \quad \text{and}$$

$$y \geq Ay + \sum_{j=1, j \neq i-n}^m C_j L_j y + C_{i-n}$$

When we rewrite the constraints in the problem (DP), using the partition of vector $x \equiv (y, z)'$ with $y \in R_+^n$ and $z \in R_+^m$, we get

$$\begin{cases} y \geq Ay + Cz, \quad \text{and} \\ z \geq Ly + e_i, \end{cases}$$

where e_i is the column m -vector whose i -th element is unity with the other elements being zero. It follows that a feasible vector of (DP) satisfies

$$y \geq Ay + \sum_{j=1, j \neq i-n}^n C_j L_j y + C_{i-n} \cdot (L_{i-n}y + 1).$$

Therefore, once we get the optimal value of (DP), x_i^* , we can compute the optimal value of (VP) as $(x_i^* - 1)/x_i^*$. By virtue of the duality theorem in linear programming, the optimal value of (LP), q_i^* is equal to x_i^* . From eq.(6), we have the desired result. \square

It is interesting to note that producing one consumption basket C_{i-n} as net output in the complete system is equivalent to producing $(L_{i-n}y + 1)$ baskets as net output in the reduced (or ordinary) system.

We can now give the definition of exploitation.

Definition 1. Labour of type i is said to be *exploited* when it is actually employed and the i -value of consumption baskets C_{i-n} obtainable in exchange for the wage rate is less than unity. When $w_i = pC_{i-n}$, the rate of exploitation for labour type i , E_i , is defined as $E_i \equiv (1 - \lambda_i^{[i]})/\lambda_i^{[i]}$.

In short, workers of type i are exploited if they rule the economy and can organize a *reproduction* plan in which they work less and at the same time they secure the present consumption C_i per unit of their service rendered.

It is convenient here to make

Definition 2. When $w_i = pC_{i-n}$ for some i , we say that workers of type i receive the *no-savings* wage rate.

We should note that no data on the size of employment of each type of labour is necessary, nor reduction to abstract labour is required. We can now establish the following fundamental theorem.

Theorem 1. If a worker of type i is employed at a wage rate not larger than the *no-savings* rate, then he/she is exploited.⁶

Proof. From eq.(1), we know that the i -value of consumption basket C_{i-n} is $\Lambda_{[i]}^{[i]} \cdot \mathbb{A}_i$, which in turn equals $\lambda_i^{[i]}$. Since $w_i \leq pC_{i-n}$ by assumption, that is, one unit of labour service of type i can buy at most one consumption basket C_{i-n} , and since $\lambda_i^{[i]} < 1$ by Proposition 1, the theorem follows. \square

It should be noted that the above theorem has nothing to do with the sizes of actual profit rates. And yet, under our Productiveness Assumption, we can show that it is impossible for all the processes to make losses at the same time. To be more precise, let us make

Definition 3. The economy is said to be in a *state of global losses* when there are a price vector, $p \in R_+^n$, and a wage rate vector, $w \in R_+^m$, such that $(p, w) \leq (p, w)\mathbb{A}$, and either one of p and w is nonzero.

⁶Propositions in this article are concerned with values only, while theorems are with values, prices and wage rates.

Note that the definition of the state of global losses implies not only losses in each industry, but also workers receive the no-savings wage rates.

Theorem 2. The Productiveness Assumption is equivalent to the impossibility of any state of global losses.

Proof. This is an immediate consequence of Corollary 3A in Tucker ([34], p.11). \square

This equivalence theorem corresponds to that between (1)' and (2)' in the main argument of Matsuo ([21], p.257). While Matsuo assumed the strict positivity of a price vector as well as the wage rate, we ask for the strict positivity in the productiveness condition as in Washida ([37]).⁷

Theorem 3. If there exists a price vector such that every industry enjoys positive profits with workers employed at their respective no-savings wage rates, the Productiveness Assumption is satisfied.⁸

Proof. A given condition is written as

$$\exists p \in R_+^n \quad \text{such that } p \gg p(A + CL).$$

From this, we can deduce

$$\exists y \in R_+^n \quad \text{such that } y \gg (A + CL)y.$$

By putting $z = Ly \in R_+^m$, we have

$$\begin{cases} y \gg Ay + Cz, \text{ and} \\ z = Ly. \end{cases}$$

Increase each element of z slightly enough so that the upper inequality system remain valid, and we obtain

$$\begin{cases} y \gg Ay + Cz, \text{ and} \\ z \gg Ly. \end{cases}$$

⁷In Matsuo ([21], p.255 and footnote 8 on p.258), Corollary 3A in Tucker ([34]) is wrongly attributed to Nikaido as Nikaido lemma. This result may date back to a still earlier time. See for example Ville ([35]).

⁸This theorem does not hold in general for models with joint production.

Put $x \equiv (y, z)'$, and we find an x which satisfies the inequality required in the Productiveness Assumption. \square

When technical progress or some social changes cause one equality in eq.(1) to an inequality, the i -values also change. We have

Proposition 3. Let i be fixed. Suppose that some coefficients in \mathbb{A} change and that, under the i -values before changes, only the j -th equation in eq.(1) becomes an inequality with the right-hand side smaller than the left-hand side. Then, the i -value of commodity j decreases, while that of the other commodities remains unchanged or decreases.⁹

Proof. Since the index i is fixed, we drop the superscript i from value vectors $\Lambda^{[i]}$ and $\Lambda_{[i]}^{[i]}$. Let us denote by $\mathbb{A}^\circ, \mathbb{A}^*, \Lambda^\circ$ and Λ^* the complete matrices and the solutions to eq.(1) before and after changes respectively. That is,

$$\Lambda^\circ = \Lambda_{[i]}^\circ \cdot \mathbb{A}^\circ, \text{ and } \Lambda^* = \Lambda_{[i]}^* \cdot \mathbb{A}^* \text{ with}$$

$$\Lambda^\circ \equiv (\lambda_1^\circ, \dots, \lambda_{n+m}^\circ) \text{ and } \Lambda^* \equiv (\lambda_1^*, \dots, \lambda_{n+m}^*).$$

A given condition is

$$\Lambda^\circ > \Lambda_{[i]}^\circ \cdot \mathbb{A}^*,$$

with a strict inequality holding in the comparison of j -th element.

The solution Λ^* can be calculated iteratively by starting from an initial vector $\Lambda(0)$ such that

$$\Lambda(0) \geq \Lambda(0)_{[i]} \cdot \mathbb{A}^*, \text{ and} \tag{7}$$

$$\Lambda(t+1) = \Lambda(t)_{[i]} \cdot \mathbb{A}^* \text{ for } t = 0, 1, \dots,$$

⁹The method of proof for this propositions is similar to that in Fujimoto, Herrero and Villar ([11], [12]), which presents the extensions of the theorems in Morishima ([23], pp.14-19). So, we can assert more about relative rates of decrease, which seems, however, irrelevant here.

where $\Lambda(t)_{[i]}$ is the vector $\Lambda(t)$ whose i -th entry is changed to unity. Since \mathbb{A}^* is nonnegative, this vector sequence is decreasing, i.e.,

$$\Lambda(0)_{[i]} \geq \Lambda(1)_{[i]} \geq \Lambda(2)_{[i]} \geq \dots,$$

and bounded from below, thus converges to $\Lambda_{[i]}^*$.

Now we know the vector Λ° can serve as an initial vector $\Lambda(0)$ to obtain Λ^* in the system after changes, because it satisfies the inequality (7). At the beginning of iteration, there is a strict inequality in the comparison of j -th elements of the two sides of (7). Hence we have proved the i -value of commodity j decreases, i.e., $\lambda_j^\circ > \lambda_j^*$, and that the values of the other commodities remain unchanged or decrease. \square

Proposition 4. In Proposition 3, when $j = i$, the i -th commodity only decreases its i -value, with the i -values of the other commodities remaining fixed.

Proof. This is evident from eq.(1). \square

This proposition is related to what Marx explained in the part of production (or enlargement) of absolute surplus value (Marx ([20])). When the working day is extended, say from 8 hours to 10 without increasing total *real* wages for a worker of type i , then the i -value of labour i decreases.¹⁰

4 A Numerical Example

We take up an example in Bowles and Gintis (1977) to show how our approach works. The data therein given as Table 1 in Bowles

¹⁰Keeping the real wages fixed requires a complicated process of price changes as well as the money wage rates.

and Gintis ([1], p.182) are:

$$A \equiv \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.2 & 0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L \equiv \begin{pmatrix} 0.07 & 0.03 & 0.1 & 1.0 \\ 0.42 & 0.09 & 0.4 & 0.7 \\ 0.21 & 0.18 & 0.7 & 0.4 \end{pmatrix}, \text{ and}$$

$$C \equiv \begin{pmatrix} 0.25 & 0.1 & 0 \\ 0 & 0 & 0 \\ 1.0 & 0.5 & 0.25 \\ 0 & 0 & 0 \end{pmatrix}$$

Bowles and Gintis ([1], p.182) named the four commodities “Food”, “Steel”, “Housing” and “Mercedes”, and three types of labour services “Supervisory”, “Primary” and “Secondary”.

Their labour values of commodities (Table 2 in Bowles and Gintis ([1], p.183)) are

$$\Lambda \equiv L(I - A)^{-1} \doteq \begin{pmatrix} 0.096 & 0.0547 & 0.140 & 1.04 \\ 0.54 & 0.22 & 0.606 & 0.874 \\ 0.324 & 0.272 & 0.852 & 0.546 \end{pmatrix} : \text{Table 2}$$

Their labour contents within each type of labour (Table 3 in Bowles and Gintis ([1], p.184)) are

$$H \equiv L(I - A)^{-1}C \doteq \begin{pmatrix} 0.164 & 0.0795 & 0.0349 \\ 0.741 & 0.357 & 0.1525 \\ 0.934 & 0.458 & 0.2139 \end{pmatrix}.$$

The Frobenius root of this matrix is $\mu(H) = 0.723 < 1$. Thus, as Potron ([31]) and Krause ([18]) proved, the equilibrium rate of profit is positive if the wage rate of each type of labour is exactly equal to the money value of its consumption basket.

To be persuasive enough to the reader, let us here compute the ‘correct’ labour values in terms of “Secondary Labour” based upon ‘conventional’ method *a la* von Neumann. First we create the augmented input coefficient matrix A^+ which includes in each process the necessary material inputs to employ “Supervisory” and “Primary” labour, i.e.,

$$\begin{aligned}
 A^+ &\equiv A + C_{(3)}L_{(3)} \\
 &= \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.2 & 0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\quad + \begin{pmatrix} 0.25 & 0.1 \\ 0 & 0 \\ 1.0 & 0.5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.07 & 0.03 & 0.1 & 1.0 \\ 0.42 & 0.09 & 0.4 & 0.7 \end{pmatrix} \\
 &= \begin{pmatrix} 0.1595 & 0.2165 & 0.365 & 0.52 \\ 0.3 & 0.1 & 0.2 & 0.3 \\ 0.28 & 0.075 & 0.3 & 1.35 \\ 0 & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

where $C_{(3)}$ and $L_{(3)}$ are the matrices C and L with the 3rd column and the 3rd row removed respectively. Then, the traditional way to calculate the labour values of normal commodities is by

$$\begin{aligned}
 \Lambda_c^{[7]} &\equiv L_3 \cdot (I - A^+)^{-1} & (8) \\
 &= \begin{pmatrix} 0.21 & 0.18 & 0.7 & 0.4 \end{pmatrix} \cdot (I - A^+)^{-1} \\
 &\doteq \begin{pmatrix} 1.03, & 0.590, & 1.70, & 3.41 \end{pmatrix},
 \end{aligned}$$

where L_3 is the 3rd row of L . It is clear that the ‘correct’ values are greater than those obtained by Bowles and Gintis, the 3rd row of Table 2 above.

Now let us proceed to our own way. The complete matrix \mathbb{A} becomes

$$\mathbb{A} \equiv \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.2 & 0.25 & 0.1 & 0 \\ 0.3 & 0.1 & 0.2 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.07 & 0.03 & 0.1 & 1.0 & 0 & 0 & 0 \\ 0.42 & 0.09 & 0.4 & 0.7 & 0 & 0 & 0 \\ 0.21 & 0.18 & 0.7 & 0.4 & 0 & 0 & 0 \end{pmatrix}.$$

Our Productiveness Assumption is satisfied by

$$x = (12, 10, 18, 1, 5, 15, 19)',$$

where a prime stands for transposition of the vector.

The solution q to eq.(4), with $i = 5$ (“Supervisory” labour), $i = 6$ (“Primary” labour), and $i = 7$ (“Secondary” labour) respectively, are approximately

$$(0.348, 0.184, 0.502, 1.964, 1.589, 0.286, 0.126),$$

$$(1.662, 0.796, 2.221, 5.024, 2.637, 2.277, 0.555), \text{ and}$$

$$(1.791, 1.027, 2.970, 5.946, 3.418, 1.664, 1.742).$$

Hence, by use of eq.(6) we compute the labour values in terms of labour type $i = 5, 6$ and 7 , respectively, as

$$\Lambda^{[5]} \doteq (0.219, 0.116, 0.316, 1.24, 0.371, 0.180, 0.0793),$$

$$\Lambda^{[6]} \doteq (0.730, 0.350, 0.975, 2.21, 1.16, 0.561, 0.244), \text{ and}$$

$$\Lambda^{[7]} \doteq (1.03, 0.590, 1.70, 3.41, 1.96, 0.955, 0.426).$$

We notice at once that the values of normal commodities in terms of “Secondary” labour coincide with those computed above in the conventional way, i.e., eq.(8). This is natural enough because of our definition eq.(1). It can be seen that all the labour values of commodities turn out larger than those in Table 2 by Bowles and Gintis ([1], p.183), because they neglected necessary labour via the inputs of other types of labour.¹¹ Incidentally, if we adopt the method in Fujimoto and Opocher ([14]), to tell if a type of labour is more skilled than another, “Supervisory” labour is the most skilled and the “Secondary” labour the least.

When each type of labour receives their respective no-savings wage rate, we can calculate the rates of exploitation as

$$E_5 = \frac{1 - 0.371}{0.371} \doteq 1.70,$$

$$E_6 = \frac{1 - 0.561}{0.561} \doteq 0.783, \text{ and}$$

$$E_7 = \frac{1 - 0.426}{0.426} \doteq 1.35.$$

Here also, we have strikingly different results from those by Bowles and Gintis ([1], p.184), in which the rate of exploitation for “Supervisory” labour is negative, i.e., -46% : that rate here is positive and 170% , the largest among three rates. This discrepancy comes from their arbitrary weights, actually equal weights, given to labour

¹¹Based upon the matrix H , Bowles and Gintis ([1]) gave two definitions of exploitation rate for each labour type. Both are, however, either inadequate or arbitrary as are criticized by Catephores ([6]). In short, they had to make up for the underestimation of values.

types when converting labour hours to some common unit. If we look at the values closely, the labour contents of housing (commodity 3) in terms of “Supervisory” labour is as low as 0.316, while those in terms of “Primary” labour and “Secondary” labour are 0.975 and 1.70, respectively: the latter are much greater. So, the value of “Supervisory” labour in terms of itself is small, though one unit of “Supervisory” labour consumes housing as large as 1.0 unit.

The reader is reminded again that in order to calculate the rates of exploitation we need no data on how many workers of each type are employed, nor conversion of various types of labour to abstract labour (or a common unit).

At the end of this section, let us give an example of a price vector under which three out of four industries incur a loss. Let a price vector be given as

$$p = (10, 2, 5, 5).$$

Then, the no-savings wage rates becomes

$$w = pC = (7.5, 3.5, 1.25).$$

Now the costs of industries are

$$p(A + CL) \doteq (3.86, 2.97, 6.43, 13.1).$$

Thus, only the first (food) industry makes profits.

5 Concluding Remarks

Remark 5.1. In proving his Fundamental Marxian Theorem, Okishio ([27], [29]) started from the the existence of a price vector, $p \in R_+^n$, such that the inequality $p \gg p(A+CL)$ holds, i.e., positive profits in every industry, then he proved that $\Lambda C < 1$. In a simple

Leontief model without joint production, this inequality implies the existence of an activity vector, $x \in R_+^n$, such that $x \gg (A + CL)x$, which in turn means our Productiveness Assumption is satisfied. Is it not better to start from the latter quantitative condition without touching upon prices and profits? In a sense, FMT's as formulated by Okishio ([27], [28], [29]) and as extended by his students Washida ([37]) and Matsuo ([21]) have been too shy to disclose the characteristics of capitalist systems, lingering on profit rates and sticking to equivalence relations. Especially, in their model with joint production, the reason why the inequality $\lambda_i^{[i]} < 1$ implies the exploitation is left unexplained, because our Proposition 2 cannot be established in the approach adopted in Washida ([37]) and Matsuo ([21]). Our method in Fujimoto and Opocher ([14]) can do this, because it depends on optimization problems. Besides, their asymmetric treatment between commodities and labour services cannot allow us to generalize the results, in a straightforward way, to models with heterogeneous labour. With the symmetric treatment, we can show the existence of exploitation of each type of labour, irrespective of profit rates of various processes. The key assumption is concerned with the size of the wage rates paid to individual types of labour.

In the main argument of Matsuo ([21], p.257), the single implication from (2)' to (3)' seems to be enough to show the essence of capitalist systems.¹² Since the target is condition (3)', his equivalence results involving the utility functions, i.e., the conditions (4)' to (5)', are redundant. In fact, he has tamed down the counter-example by Petri ([30]), not through the introduction of utility, but simply by postulating strict positivity of values. (See Definition 3 in Matsuo ([21], p.254)).

Hence, we may add that the so-called Commodity Exploitation Theorem has come down, not to demolish the FMT, but to induct

¹²Thus, our Theorem 2 above is not essential, either.

it into the Hall of Fame.¹³

Remark 5.2. Since the first definition of exploitation rates in Bowles and Gintis ([1]) is not adequate, their Theorem 2' (Bowles and Gintis ([2], p.312) and Mori ([22], p.527)) is confused. For example, they wished to show that if the equilibrium rate of profit is positive, there is at least one type of workers who are exploited. In our framework, however, the positive equilibrium profit rate implies the Productiveness Assumption (Theorem 3 above), and so every type is exploited.

Remark 5.3. The method of handling heterogeneous labour by Okishio ([28], [29]), which is adopted also by Morishima (1973), requires the life-long data on how various types of labour are created or maintained, and each type of labour should be formed in a unique way. After all, the Okishio's method can work only in models without joint production.

Remark 5.4. Fujimori's approach ([7]) to the abstract labour is somewhat similar to Okishio's ([28]). He uses, however, the actual employment data of various labour types, thus the method can be affected by those data which are not technological nor biological. See also Hollander ([15]).

Remark 5.5. The commodity content of a commodity by Manresa, Sancho and Vegara ([19]) is equivalent to ours when our values are measured in terms of commodity $i \leq n$. To prove the equivalence, one may use the Banachiewicz identity concerning the inverse of a partitioned square matrix. See Fujimoto, Hisamatsu and Ranade ([13]).

¹³The Commodity Exploitation Theorem has shown a theorem like our Proposition 1 for any commodity by treating commodities and homogeneous labour in a symmetrical way. For a criticism levelled against this theorem, see Fujimoto and Fujita ([10]).

Remark 5.6. A view on exploitation among heterogeneous labour, which has been discussed in Bowles and Gintis ([1], [2], and [4]), is incorporated in Fujimoto ([9]).

Remark 5.7. Thanks to von Neumann ([36]) and Morishima ([24], [25]), we can generalize our results in this paper, except for Theorem 3 and Propositions 3 and 4, to models with joint production by use of the method introduced in Fujimoto and Opocher ([14]). In so doing, we can at the same time allow for alternative consumption baskets for reproduction of each type of labour service, durable consumption goods, use of labour in reproducing labour service, and even joint production of labour services like in a family.¹⁴ In this article we focus on the case of single production for the sake of readability.

Remark 5.8. We have avoided to introduce the indecomposability of \mathbb{A} , nor the semi-positivity of L . Thus, for example, the case of $L = 0$, i.e., a fully automatic system, is also allowed for. It is easy to verify that all our propositions and Theorem 2 remain good in such a system also. Theorem 1 and 3 are *logically* valid because their respective protasis is null.

Remark 5.9. Our title carries the word “capitalist”. Actually all we require for the system under our analysis is the economy-wide stability of various coefficients. This sort of stability has been observed so far only capitalist and socialist systems, we suppose. One more point is that in our economy workers are supposed to be alien to the decision making about what to produce, how to produce, how much to produce, and how much they get as their wages. To extract one more feature of capitalist systems, we had better include the unemployed in the ‘exploited’ labour, because

¹⁴A family seems to be an efficient coalition to reduce values, though it may not always be a clever formation to lessen the costs of living.

they are necessary to keep down wage rates.

References

- [1] Bowles S., Gintis H. (1977): ‘The Marxian theory of value and heterogeneous labour: a critique and reformulation’, *Cambridge Journal of Economics*, 1, pp.173-192.
- [2] Bowles S., Gintis H. (1978): ‘Professor Morishima on heterogeneous labour and Marxian value theory’, *Cambridge Journal of Economics*, 2, pp.311-314.
- [3] Bowles S., Gintis H. (1981a): ‘Labour heterogeneity and the labour theory of value: a reply’, *Cambridge Journal of Economics*, 5, pp.285-288.
- [4] Bowles S., Gintis H. (1981b): ‘Structure and practice in the labor theory of value’, *Review of Radical Political Economy*, 12, pp.1-26.
- [5] Bródy A. (1970): *Proportions, Prices and Planning: A Mathematical Restatement of the Labor Theory of Value*, Akadémiai Kiadó, Budapest.
- [6] Catephores G. (1981): ‘On heterogeneous labour and the labour theory of value’, *Cambridge Journal of Economics*, 5, pp.273-280.
- [7] Fujimori Y. (1982): *Modern Analysis of Value Theory*, Springer-Verlag, Berlin.
- [8] Fujimoto T.(1978): ‘Exploitation, profits, and growth: a disequilibrium analysis’, *Economic Studies Quarterly*, 29, pp.268-275.

- [9] Fujimoto T.(2009): ‘The concept of exploitation in a general linear model with heterogeneous labour’, *Investigación Económica* (UNAM), 68, pp.51-82.
- [10] Fujimoto T., Fujita Y. (2008): ‘A refutation of commodity exploitation theorem’, *Metroeconomica*, 59, pp.530-540.
- [11] Fujimoto T., Herrero C., Villar A. (1983): ‘Technical changes and their effects on the price structure’, *Metroeconomica*, 35, pp.123-127.
- [12] Fujimoto T., Herrero C., Villar A. (1985): ‘A sensitivity analysis for linear systems involving M-matrices and its application to the Leontief model’, *Linear Algebra and Its Applications*, 64, pp.85-91.
- [13] Fujimoto T., Hisamatsu H., Ranade R.(2004): ‘The Schur complements and the Le Chatelier-Braun principle’, *Kagawa University Economic Review*, 77, pp.389-397.
- [14] Fujimoto T., Opocher A. (2007): ‘Commodity content in a general input-output model’, to appear in *Metroeconomica*.
- [15] Hollander H. (1978): ‘A note on heterogeneous labour and exploitation’, *Discussionsbeiträge zur Politischen Ökonomie*, 14.
- [16] Krause U. (1979): *Geld und abstrakte Arbeit. Über die analytischen Grundlagen der politischen Ökonomie*, Campus, Frankfurt am Main.
- [17] Krause U. (1980): ‘Abstract labour in general joint systems’, *Metroeconomica*, 32, pp.115-135.
- [18] Krause U. (1981): ‘Heterogeneous labour and the fundamental Marxian theorem’, *Review of Economic Studies*, 48, pp.173-178.

- [19] Manresa A., Sancho F., Vegara J. M.(1998): ‘Measuring commodities’ commodity content”, *Economic Systems Research*, 10, pp.357-365.
- [20] Marx K.(1867): *Capital I*, The original German edition published in 1867.
<http://www.marxists.org/archive/marx/works/download/capita>
- [21] Matsuo T. (2008): ‘Profit, surplus product, exploitation, and less than maximized utility: a new equivalence proposition on the fundamental Marxian theorem’, *Metroeconomica*, 59, pp.249-265
- [22] Mori K. (2008): ‘Maurice Potron’s linear economic model: a *de facto* proof of “fundamental Marxian theorem” ’, *Metroeconomica*, 59, pp.511-529.
- [23] Morishima M. (1964): *Equilibrium, Stability and Growth: A Multi-sectoral Analysis*, Oxford University Press, London.
- [24] Morishima M. (1973): *Marx’s Economics: A Dual Theory of Value and Growth*, Cambridge University Press, Cambridge.
- [25] Morishima M. (1974): ‘Marx in the light of modern economics’, *Econometrica*, 42, pp.611-632.
- [26] Morishima M. (1978): ‘S. Bowles and H. Gintis on the Marxian theory of value and heterogeneous labour’, *Cambridge Journal of Economics*, 2, pp.305-309.
- [27] Okishio N. (1963): ‘Mathematical note on Marxian theorems’, *Weltwirtschaftliches Archiv*, 91, pp.287-299.
- [28] Okishio N. (1965): *The Fundamental Theory of Capitalist Economy*, (in Japanese), Soubunsha, Tokyo.

- [29] Okishio N. (1977): *Marx's Economics: Values and Prices*, (in Japanese), Chikuma-Shobou, Tokyo.
- [30] Petri F. (1980): 'Positive profits without exploitation: a note on the generalized fundamental Marxian theorem', *Econometrica*, 48, pp. 531-533.
- [31] Potron M. (1913): 'Quelques propriétés des substitutions linéaires à coefficients ≥ 0 et leur application aux problèmes de la production et des salaires', *Annales Scientifiques de l'É.N.S.* 3e série, t.30, pp.53-76.
- [32] Steedman I. (1977): *Marx after Sraffa*, New Left Books, London.
- [33] Steedman I. (1980): 'Heterogeneous labour and «classical» theory', *Metroeconomica*, 32, pp.39-50.
- [34] Tucker A.W. (1956): 'Dual systems of homogeneous linear relations' in Kuhn H.W. and Tucker A.W.(eds.): *Linear Inequalities and Related Systems*, Princeton University Press, New Jersey. pp.3-18.
- [35] Ville J. (1938): 'Sur la théorie générale des jeux où intervient l'habileté des Joueurs', *Traité du Calcul des Probabilités et des Ses Applications*, 2, Part 2, pp.105-113.
- [36] von Neumann J. (1945-46): 'A model of economic equilibrium', *Review of Economic Studies*, 13, pp.1-9.
- [37] Washida T. (1988): 'Reconstruction of the generalized fundamental Marxian theorem', *Economic Studies Quarterly*, 39, pp.277-283.