

Labour Values: An Exposition

Takao Fujimoto*

Abstract This note is to explain a portion of the paper, Fujimoto and Opocher [8], especially the new definition of labour values in an easy and simple way. Our model allows for joint production as well as heterogeneous labour.

1 Introduction

As we proceed along the approach adopted by the classical economists, that is, to determine the values of various commodities before their prices come on to the stage without resorting to supply and demand, there arise difficulties brought in by joint production, heterogeneous labour, international trade, exhaustible resources etc. Morishima [17, 18] proposed a solution against the existence of joint production, using a linear programming problem. Bowles and Gintis [1] offered a definition in a model with heterogeneous labour but without joint production. On the other hand, several authors considered how to grasp the amount of a particular commodity contained in various products directly and indirectly: see Jeong [11, 12], Fujita [9], and Manresa et al. [14]. These contributions are all for models with single production and homogeneous labour.

Our method in Fujimoto and Opocher [8] can include all the above as a special case. A new solution therein is for a model with joint production, heterogeneous labour and possible international

*Faculty of Economics, Fukuoka University, Fukuoka, Japan

trade. In this note, we explain the new definition in a model with joint production as well as heterogeneous labour, concentrating on a closed economy. In section 2, we derive our definition starting from a Leontief-type model. Section 3 is devoted to a full-fledged version. The final section 4 contains some remarks.

2 A Leontief Model

Let us consider a Leontief-type simple input-output model without joint production and with homogeneous labour. There are n kinds of commodities, and corresponding to these, we have n industries. The symbols are:

- A : a given $n \times n$ matrix of material input coefficients,
- ℓ : a given row n -vector of labour input coefficients,
- c : a given column n -vector of consumption basket
to reproduce one unit of labour force,
- I : the identity $n \times n$ matrix,
- λ : the row n vector of labour values, and
- λ_ℓ : the labour value of one unit of labour force.

As explained in Okishio [20] and Morishima [17], the labour value vector λ is defined by

$$\lambda = \lambda A + \ell,$$

and the labour value of labour is computed by

$$\lambda_\ell = \lambda \cdot c.$$

Now we define an $(n + 1) \times (n + 1)$ matrix \mathbb{A} as¹

$$\mathbb{A} \equiv \begin{pmatrix} A & c \\ \ell & 0 \end{pmatrix}.$$

¹This enlarged matrix is called the ‘complete matrix’ in Bródy [3].

Then above two equations can be combined into one, leading to

$$\left(\lambda, \lambda_\ell \right) = \left(\lambda, 1 \right) \mathbb{A}. \quad (1)$$

It is important to note here that in eq.(1), the *direct* input of labour is given unity as its value on the RHS, while the value of one unit of labour is calculated as λ_ℓ on the LHS. Then eq.(1) is rewritten as

$$\left(\lambda, 1 \right) = \left(\lambda, 1 \right) \mathbb{A} + \left(\mathbf{0}'_n, 1 - \lambda_\ell \right), \quad (2)$$

where $\mathbf{0}_n$ is the column n -vector whose elements are all zero with a prime indicating transposition. This leads to

$$\left(v\lambda, v \right) = \left(v\lambda, v \right) \mathbb{A} + \left(\mathbf{0}'_n, v(1 - \lambda_\ell) \right),$$

where v is a positive scalar. Supposing $(1 - \lambda_\ell) > 0$, we normalize v so that $v(1 - \lambda_\ell) = 1$. By putting $q \equiv \left(v\lambda, v \right)$, we finally obtain the equation

$$q = q \cdot \mathbb{A} + \left(\mathbf{0}'_n, 1 \right). \quad (3)$$

We make the following assumption.

Assumption AP (Productiveness assumption): There exists a nonnegative column $(n + 1)$ -vector x such that $x \gg \mathbb{A}x$. Thus, $(I - \mathbb{A})^{-1} > 0$.²

Now eq.(3) is solved for q as

$$q = \left(\mathbf{0}'_n, 1 \right) (I - \mathbb{A})^{-1}. \quad (4)$$

On the other hand, from eq.(2) we get

$$\left(\lambda, 1 \right) = \left(\mathbf{0}'_n, 1 - \lambda_\ell \right) (I - \mathbb{A})^{-1}.$$

It is evident from eq.(4) that q is the last row of the inverse $(I - \mathbb{A})^{-1}$, thus the above equation gives

²See Hawkins and Simon [10] for their conditions concerning A .

$$1 = q_{n+1}(1 - \lambda_\ell), \text{ that is, } \lambda_\ell = \frac{q_{n+1} - 1}{q_{n+1}}, \text{ and} \quad (5)$$

$$\lambda_j = \frac{q_j}{q_{n+1}} \text{ for } j = 1, \dots, n. \quad (6)$$

It naturally follows that $0 \leq \lambda_\ell < 1$ and $\lambda_j \geq 0$ for $j = 1, \dots, n$, i.e., the labour value of one unit of labour is less than one.³ When labour is indispensable to produce a basket c , we have $0 < \lambda_\ell$. Therefore, when the productiveness assumption AP is given, all we have to do is to solve eq.(4) first, and then calculate labour values using eqs.(5) and (6).

Through the studies on nonsubstitution theorems, we know the solution q can be obtained by solving the following linear programming problem.⁴

$$\max q_{n+1} \text{ subject to } q \leq q \cdot \mathbb{A} + \left(\mathbf{0}'_n, 1 \right) \text{ and } q \geq \mathbf{0}'_n. \quad (7)$$

The dual problem to the above is

$$\min x_{n+1} \text{ subject to } x \geq \mathbb{A} \cdot x + \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix} \text{ and } x \geq \mathbf{0}_n.$$

In fact these linear programming problems have the optimal solutions by which the constraints are all satisfied with the strict equality. The meaning of the constraints in the primal problem is that in each process the value of output cannot exceed the total value of inputs, and in the dual problem the constraints require that the gross output vector x should produce one unit of labour force as the net output.

³This has nothing to do with exploitation. See Fujimoto and Fujita [6]. See also Okishio [19], Roemer [21, 22, 23], and Bowles [2].

⁴For nonsubstitution theorems, see, e.g., Fujimoto et al. [7].

3 A Generalized Morishima-von Neumann Model

Now we are ready to jump at a general von Neumann model with heterogeneous labour, in which there can be alternative household activities to reproduce each type of labour. Besides, there can be durable consumption goods as well. In more details, our model is a generalization of Morishima model (Morishima [16]). That is, different from the original model in von Neumann [26], we explicitly deal with labour input coefficients, and moreover we allow for the existence of joint production as well as heterogeneous labour. Thus, we are able to deal with durable capital goods. Various types of labour are treated exactly like normal commodities, and so we use the symbol \mathbb{B} and \mathbb{A} as the output and input coefficient matrices, both of which now have n rows and m columns. There are altogether n kinds of goods, services, and various types of labour. On the other hand, there exist m production processes or household activities. This way of formulation enables us to take into consideration durable consumption commodities in household activities: a durable consumption commodity in a column of household activity of \mathbb{A} will appear in the corresponding column of \mathbb{B} as one period, say one year, older commodity. For each type of labour, there can be more than one household activity to reproduce that labour. Workers may save a part of their incomes, and may have properties. These complicating elements from the real world do not disturb our study while we deal with values and exploitation: this should be true in any linear model including Leontief models.

Now we choose a type of labour as the standard and let it be the i -th labour commodity. We give our definition of labour values as follows.

Definition of values for our general model: Values in a general input-output model are nonnegative magnitudes assigned to commodities (including services and various types of labour) such

that the value of the standard commodity be maximized under the condition that the total value of the output of each possible process should not exceed that of the input. When calculating the total value of the input of a process, unity is assigned to the direct input of the standard commodity.⁵

Our productiveness assumption here is:

Assumption APG: There exists an $x \geq \mathbf{0}_m$ such that

$$(\mathbb{B} - \mathbb{A})x \gg \mathbf{0}_n.$$

Having defined values as above, we can now explain how to compute values in a way similar to the problem (7) in section 2. Let us first define the following vectors:

$$\begin{aligned} \Lambda^{[i]} &\equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \dots, \lambda_{i-1}^{[i]}, \lambda_i^{[i]}, \lambda_{i+1}^{[i]}, \dots, \lambda_n^{[i]}), \text{ and} \\ \Lambda_{[i]}^{[i]} &\equiv (\lambda_1^{[i]}, \lambda_2^{[i]}, \dots, \lambda_{i-1}^{[i]}, 1, \lambda_{i+1}^{[i]}, \dots, \lambda_n^{[i]}). \end{aligned}$$

The vector $\Lambda^{[i]}$ is the vector of values with i -th labour being the standard of value, and the element $\lambda_j^{[i]}$ stands for the value of commodity j with i -commodity as the standard of value. Our definition above is rewritten like this:

Findout $\Lambda^{[i]} \geq 0$ such that $\lambda_i^{[i]}$ should be maximized

$$\text{subject to } \Lambda^{[i]} \cdot \mathbb{B} \leq \Lambda_{[i]}^{[i]} \cdot \mathbb{A}.$$

We can proceed as we have done for a Leontief model in section 2. That is, the constraint in this problem can be transformed first

⁵Among a plural number of solutions, we adopt those which realize the maximum number of equalities in the constraints. And yet, a solution may not be unique.

through adding $(1 - \lambda_i^{[i]}) \cdot b^{(i)}$ to both sides, then multiplying both sides by nonzero nonnegative v , yielding

$$v \cdot \Lambda_{[i]}^{[i]} \cdot \mathbb{B} \leq v \cdot \Lambda_{[i]}^{[i]} \cdot \mathbb{A} + v \cdot (1 - \lambda_i^{[i]}) \cdot b^{(i)},$$

where $b^{(i)}$ is the i -th row of \mathbb{B} . Then, we set

$$v \cdot (1 - \lambda_i^{[i]}) = 1 \quad \text{or} \quad \lambda_i^{[i]} = 1 - \frac{1}{v}.$$

This normalization yields as our constraint

$$v \cdot \Lambda_{[i]}^{[i]} \cdot \mathbb{B} \leq v \cdot \Lambda_{[i]}^{[i]} \cdot \mathbb{A} + b^{(i)}.$$

Since we have $\lambda_i^{[i]} = 1 - 1/v$ from our normalization, maximizing v is equivalent to maximizing $\lambda_i^{[i]}$. Writing $v \cdot \Lambda_{[i]}^{[i]}$ simply as a variable vector q , thus $q_i \equiv v$, we have the linear programming problem (DG) analogous to (7):

$$(DG) \quad \max q_i \quad \text{subject to} \quad q' \mathbb{B} \leq q' \mathbb{A} + b^{(i)} \quad \text{and} \quad q' \geq \mathbf{0}'_n.$$

We first solve this linear programming problem. Next, the values can be calculated exactly as in eqs.(5) and (6), i.e.,

$$\lambda_i^{[i]} = \frac{q_i^* - 1}{q_i^*} \quad \text{and} \quad \lambda_j^{[i]} = q_j^*/q_i^* \quad \text{for} \quad j = 1, \dots, n, \quad j \neq i. \quad (8)$$

It is not difficult to establish

Proposition. Given the productiveness assumption (APG), the i -th labour value of i -th labour is less than unity.

Proof. Consider the linear programming problem dual to the above problem (DG) :

$$(PG) \quad \min b^{(i)} x \quad \text{subject to} \quad \mathbb{B}x \geq \mathbb{A}x + e_{[i]} \quad \text{and} \quad x \geq \mathbf{0}_m,$$

where $e_{[i]}$ is the n -column vector whose i -th entry is unity with all the remaining elements being zero. By the duality theorem, we know that the optimal values satisfy $q_i^* = b^{(i)}x^*$. On the other hand, it is clear from the constraint in (PG) that $b^{(i)}x^* \geq 1$. Thus we get $q_i^* \geq 1$, which gives $0 \leq \lambda_i^{[i]} < 1$ because of eq.(8). \square

4 Remarks

4.1. Smith [24] emphasized the importance of division of labour, which, in an advanced capitalist economy, can be a synonym of heterogeneity of labour. Abstract labour by Marx [15] seems to be a teleological concept to construct a two-class model, and then a theory of exploitation. See Steedman [25] and Fujimoto [5].

4.2. As our formulation in section 3 is symmetrical between labour types and commodities, we can easily define the values in terms of a commodity chosen as the standard. Thus, our definition so modified is more general than those in Fujita [9] and Manresa [14].

4.3. It is easy to notice that our definition includes the definition by Morishima [18] and that by Bowles and Gintis [1] as a special case. For example, in Morishima’s model, our complete matrices are:

$$\mathbb{B} \equiv \begin{pmatrix} B & \mathbf{0}_{n-1} \\ \mathbf{0}'_{m-1} & 1 \end{pmatrix}, \text{ and } \mathbb{A} \equiv \begin{pmatrix} A & c \\ \ell & 0 \end{pmatrix},$$

where B and A are normal von Neumann material output and input coefficient matrices, and the final n -th row stands for the homogeneous labour. Then, the constraints in our linear programming problem (DG), after dividing the both sides by q_n , becomes

$$\begin{aligned} \Lambda B &\leq \Lambda A + \ell, \text{ and} \\ 1 &\leq \Lambda c + \frac{1}{q_n}. \end{aligned}$$

Here Λ is $\Lambda^{[n]}$ in the previous section, i.e., the labour value vector. Maximizing q_n is equivalent to maximize Λc . Thus, we reach the same definition as in Morishima [18]. In his model, there are no durable consumption goods, nor alternative household activities to reproduce the homogeneous labour. These restrictions are removed in our formulation.

4.4. Krause [13] and Fujimori [4] considered the models with heterogeneous labour. They, however, adopted a method of reducing various types of labour to a particular labour, which is unnecessary in our definition.

4.5. One attempt is made in Fujimoto and Opocher [8] to define labour values in a model of open economies.

References

- [1] Bowles S., Gintis H. (1977): “The Marxian theory of value and heterogeneous labour: a critique and reformulation”, *Cambridge Journal of Economics*, 2, pp.173-192.
- [2] Bowles S., Gintis H. (1981): “Structure and practice in the labor theory of value”, *Review of Radical Political Economy*, 12, pp.1-26.
- [3] Bródy A. (1970): *Proportions, Prices and Planning: A Mathematical Restatement of the Labor Theory of Value*, Akadémiai Kiadó, Budapest.
- [4] Fujimori Y. (1982): *Modern Analysis of Value Theory*, Springer-Verlag, Berlin.
- [5] Fujimoto T.(1978): “Exploitation, profits, and growth: a disequilibrium analysis”, *Economic Studies Quarterly*, 29, pp.268-275.
- [6] Fujimoto T., Fujita Y. (2006): “A refutation of commodity exploitation theorem”, Working Paper, Fukuoka University, to appear in *Metroeconomica*.

- [7] Fujimoto T., Herrero C., Ranade R.R., Silva J.A., Villar A. (2003): “A complete characterization of economies with the nonsubstitution property”, *Economic Issues*, 8, pp.63-70.
- [8] Fujimoto T., Opocher A. (2007): “Values and prices in a general input-output model”, *mimeo*, University of Padova.
- [9] Fujita Y. (1991): “A further note on a correct economic interpretation of the Hawkins-Simon conditions”, *Journal of Macroeconomics*, 13, pp.381-384.
- [10] Hawkins D., Simon H. A. (1949): “Note: some conditions of macroeconomic stability”, *Econometrica*, 17, pp.245-248.
- [11] Jeong K. (1982): “Direct and indirect requirements: A correct economic interpretation of the Hawkins-Simon conditions”, *Journal of Macroeconomics*, 4, pp.349-356.
- [12] Jeong K. (1984): “The relation between two different notions of direct and indirect input requirements”, *Journal of Macroeconomics*, 6, pp.473-476.
- [13] Krause, U. (1981): “Heterogeneous Labour and the Fundamental Marxian Theorem”, *Review of Economic Studies*, 48, pp.173-178.
- [14] Manresa A., Sancho F., Vegara J. M.(1998): “Measuring commodities’ commodity content”, *Economic Systems Research*, 10, pp.357-365.
- [15] Marx K. (1867): *Capital I*, available in the Internet at <http://www.marxists.org/archive/marx/works/download/capital.zip>. The original German edition was published in 1867.
- [16] Morishima M. (1964): *Equilibrium, Stability and Growth: A Multi-Sectoral Analysis*, Oxford Univ. Press, London.

- [17] Morishima M. (1973): *Marx's Economics: A Dual Theory of Value and Growth*, Cambridge University Press, Cambridge.
- [18] Morishima M. (1974): "Marx in the light of modern economics", *Econometrica*, 42, pp.611-632.
- [19] Okishio N. (1963): "Mathematical note on Marxian theorems", *Weltwirtschaftliches Archiv*, 91, pp.287-299.
- [20] Okishio N. (1977): *Marx's Economics: Values and Prices*, (in Japanese), Chikuma-Shobou, Tokyo.
- [21] Roemer J.E. (1981): *Analytical Foundations of Marxian Economic Theory*, Cambridge: Harvard University Press.
- [22] Roemer J.E. (1982): *A General Theory of Exploitation and Class*, Harvard University Press, Cambridge.
- [23] Roemer J.E. (1986): "Should Marxists be interested in exploitation?" in Roemer J.E. (ed.): *Analytical Marxism*, Cambridge University Press, Cambridge, pp.260-282.
- [24] Smith A. (1776): *An Inquiry into the Nature and Cause of the Wealth of Nations*, <http://www.gutenberg.net/>.
- [25] Steedman I. (1977): *Marx after Sraffa*, New Left Books, London.
- [26] von Neumann J. (1945-46): "A model of economic equilibrium", *Review of Economic Studies*, 13, pp.1-9.