A Proposition Dual to the Nonsubstitution Theorems*

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ABSTRACT. In this note, we give a proposition which is dual to the nonsubstitution theorems. The nonsubstitution theorems assert that whatever composition of the final demand vector is designated, the same group of production processes can remain efficient under a set of conditions. Our proposition tells us that under almost the same set of conditions, whatever labour input coefficient vector is observed, the same group of commodities remain non-free. This is an easy consequence of linear programming approach to the nonsubstitution theorems. In the literature, however, this proposition has been overlooked or neglected.

Keywords: Duality, Free Goods, Linear Programs, Inverse Positivity, Nonsubstitution Theorems

1 Introduction

This note is to present a proposition which is dual to the nonsubstitution theorems in linear economic models. The nonsubstitution theorems tell us that under certain conditions for any nonnegative vectors of final demand, we can find the same set of production processes which are efficient in terms of labour force employed, provided that a rate of balanced growth is given and fixed. At present, an efficient proof of the nonsubstitution theorems is by

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using a linear programming problem: see Bose[6], Chander[8], and Fujimoto et al.[20]. Since a duality relation may hold in linear programming problems, we can establish a proposition dual to the nonsubstitution in an easy way. That is, our new theorem asserts that whatever changes are made in the labour input coefficient vector, the same set of commodities are found to be non-free, provided that a uniform rate of profit is fixed and no process earns super-normal profits. For the sake of simplicity, we assume a given profit rate is zero. In the literature, however, this theorem has been neglected, surely because it is not so impressive as the nonsubstitution theorems.

In the next section, we explain notation and state our assumptions. Then in section 3, the main proposition is presented. The final section contains a numerical example and some remarks. In the references, we give the papers and the books relevant to the topic for the reader's convenience, though many of them are not cited in the text.

2 Notation and Assumptions

Let X and Y be the Euclidean spaces over the real field $\mathbb{R}$, and we assume X is of dimension m, i.e., $\mathbb{R}^m$, and Y is of dimension n, i.e., $\mathbb{R}^n$. The spaces X and Y have their nonnegative orthant, $X_+ \equiv \mathbb{R}^m_+$ and $Y_+ \equiv \mathbb{R}^n_+$ respectively. By these cones X_+ and Y_+, we have a natural order in X and Y. Symbols in vector comparison are:

$$
\begin{aligned}
x &\geq y \Longleftrightarrow x_i \geq y_i \text{ for all } i; \\
x &> y \Longleftrightarrow x_i \geq y_i \text{ for all } i \text{ and } x \neq y; \\
x &\gg y \Longleftrightarrow x_i > y_i \text{ for all } i.
\end{aligned}
$$

These symbols are used also for matrix comparison.

Given $n \times m$ matrices A and B stand for the material input coefficient matrix and output coefficient matrix respectively, with their corresponding columns being individual production processes. We define $M \equiv B - A$. A given n-column vector $d \gg 0$ means the fixed final demand vector, while a parametrical m-row vector $\ell \in X_+$ represents a labour input coefficient vector. The symbols $\ell \cdot x$ for $x \in X_+$ and $y \cdot d$ for $y \in Y_+$ show the inner-product. We interpret x as a variable m-column vector of activity levels of production processes, while y is a variable n-row vector of *wage-unit* prices. Thus, the inner-product $\ell \cdot x$ means the amount of labour force employed when the processes are used as described by x, and the inner-product $y \cdot d$ means the value of the final demand vector when the prevailing prices are y.

When we deal with the nonsubstitution theorems, it is assumed that $n \leq m$. In this note, we assume $n \geq m$: the number of commodities is greater that that of production processes.

Let us now consider the following linear programming problem:

$$\min_x \ \ell \cdot x \ \text{ subject to } Mx \geq d \ \text{ and } x \in X_+, \tag{P}$$

and its dual:

$$\max_y \ y \cdot d \text{ subject to } yM \leq \ell \text{ and } y \in Y_+. \tag{D}$$

We make the following assumptions.

Assumption A1. The problem (P) has a feasible vector $x^\circ \in X_+$ such that $Mx^\circ \gg 0$.

This assumption A1 requires that our economy be productive enough so that there is a vector of activities which realizes excess supply of every commodity. Thanks to this assumption, an optimal solution pair x^* and y^* exist and there is no duality gap, $\ell \cdot x^* = y^* \cdot d$.

We collect the commodities which have a positive price in y^*, and denote by NF the index set of these non-free goods. One more assumption is:

Assumption A2. The cardinality of NF is not less than m, and moreover there exists a regular $m \times m$ submatrix M^* of M such that $(M^*)^{-1} > 0$.

This assumption is rather restrictive, and yet is satisfied when each commodity is produced only by one process, which is dual to the supposition of absence of joint production. (In our example below, however, every commodity is produced by a plural number of processes.)

Now we define an m-row vector $y^{**} \equiv \ell(M^*)^{-1}$, and form an n-row competitive price vector $y^{*\circ}$ which equals y^{**} for the index set in M^* with the remaining entries being zero. Thus we have

$$y^{*\circ}M = \ell,$$

from which it follows

$$y^{*\circ}Mx^* = \ell \cdot x^*. \tag{1}$$

On the other hand, because of the rule of free goods (i.e., if there is an excess supply for a good in an optimal production x^*, that good is free in an optimal price vector y^*), and since $y^{*\circ}$ has more zero's than y^*, we have

$$y^{*\circ} M x^* = y^{*\circ} \cdot d. \tag{2}$$

Hence it is confirmed that $y^{*\circ}$ is also an optimal solution to (D). One more relation

$$M^* x^* = d \tag{3}$$

follows from the definition of M^*.

3 Main Proposition

We now prove our main proposition.
Theorem: Whatever changes are made in the labour input coefficient vector, we can find the same set of commodities which remain non-free under a competitive price vector. Besides, that competitive price vector realizes the maximum value of the fixed final demand among those price vectors which are feasible in the dual problem (D).
Proof. Whatever ℓ is given, we define y^{**} and $y^{*\circ}$ as in the previous section, and the first half of the theorem is obtained. Next, the latter half of the theorem asserts that if $yM \leq \ell$ and $y \in Y_+$, then $y \cdot d \leq y^{*\circ} \cdot d$. Postmultiply by x^* both sides of the constraint of (D), and we get

$$y \cdot d = yMx^* \leq \ell \cdot x^* = y^{*\circ} \cdot d\,,$$

where the left-hand equality comes from eq.(3) and the right-hand equality from eqs.(1) and (2). This completes the proof. □

4 An Numerical Example and Remarks

Consider the following example:

$$B \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}, \ A \equiv \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ d \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ \text{and } \ell = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

Thus,

$$M \equiv \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{pmatrix}.$$

It is not difficult to verify that

$$x^* = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ y^* = \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \ \text{and } \ell \cdot x^* = y^* \cdot d = 3.$$

We now compute

$$y^{**} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \ \text{and } y^{*\circ} = y^*,$$

and then can further calculate

$$M^* = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \ \text{and its inverse } (M^*)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

When ℓ becomes $\begin{pmatrix} 1 & 1 & 3 \end{pmatrix}$, $y^{**} = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$ and $y^{*\circ} = \begin{pmatrix} 1 & 2 & 2 & 0 \end{pmatrix}$. And we have

$$\ell \cdot x^* = y^{*\circ} \cdot d = 5.$$

As the remarks, we write on two possible generalizations. The first is to allow for some negative elements in $(M^*)^{-1}$ as was done in Fujimoto et al.[20]. In the paper [20], the existence of negative elements is associated with durable capital goods, and in this dual side also, a natural interpretation of negative elements may be the same, or more generally *proper* joint production. The second extension is in line with Fujimoto et al.[21], incorporating some kinds of externalities and variable returns.

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