Learning with ‘Learning from “Learning by Doing”’

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Abstract

This paper builds a fluid approximation to Solow’s modification to Arrow’s model for “learning by doing.” By removing randomness from Solow’s modification it becomes clear that Arrrow’s model permits various deterministic modifications. One such possibility is to accommodate the observation that an investment that is too small can be worse than no investment at all.

1 Introduction

Studies on economic growth today emphasize progress in technology as the main driving force of economic growth [6, 2]. For the historical context in which technology came to be considered more important than other factors such as population growth or availability of natural resources see [3], which is a part of [4].

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In "Learning by Doing" [1] Arrow proposes a model of technological *improvement*, generally considered a precursor of endogenous growth models, which is schematically

$$\text{Progress} = \text{Improvement (= Learning by doing)}.$$

In 'Learning from "Learning by doing"' [9] Solow – the originator of the modern theory of exogenous economic growth [8] – elaborates Arrow's model to include technological *innovation*, which is exogenous and stochastic, in addition to the technological improvement:

$$\text{Progress} = \text{Innovation} + \text{Improvement}.$$

The present article is a step in learning with 'Learning from "Learning by doing"' comprising:

1. a simulation program of the growth model,
2. a fluid approximation to Solow's model, and
3. a modification to Arrow's model to describe that too little investment may produce no technological progress.

Arrow's model of improvement is deterministic while the innovation portion of Solow's model is stochastic. The stochastic nature makes it difficult for the reader to grasp how Solow's model works; it even deprives the existence of a steady state exponential growth. The main tool Solow uses to investigate his model is therefore simulation. In order to examine the stochastic aspects of the model in depth he adopts a simplified version of the model of technological progress such that the cumulative gross investment is exogenous. Since a major attraction of both Arrow's and Solow's models for most readers is that the investment is endogenous, a simulation model would be desirable which embodies Solow's model in its entirety.

It soon becomes apparent, though, that the model cannot easily be understood even with the help of such a simulator because of the stochastic behavior it exhibits. The remedy is to simplify the model by stripping it of random variables. Fluid approximation is used to that end.

Once the randomness is out, it is easy to compare Arrow's and Solow's models. Their comparison reveals a possiblility to modify Arrow's model in ways different from Solow's. One such example is suggested, which describes that a too small investment can be worse than no investment.

The remainder of the paper is organized as follows. Sections 2 and 3 breifly describe Arrow's and Solow's models with Appendixes A and B containing corresponding computer programs. A fluid approximation to Solow's model is developed in section 4. A possible modification to Arrow's model is presented in section 5. The results are summarized in section 6.

2 Arrow's model

The symbols used in Arrow's model are as follows. The page numbers are for [9]; those without are not explicitly contained therein.

$t=1,2,\ldots$	Time	p.6
$0<g(t)=:g$	Rate of investment	p.6
$0<G(t)=:G$	Cumulative investment	p.6
$0<L$	Labor	p.7
$0<n\approx 1/3\leq 1$	"Learning curve"	p.6
$0<a<1$	Constant	p.7
$0<b<1$	Constant	p.6

Let $0 < bg^{-n}$ be the additional labor needed to operate investment g, which means that improvement is labor saving: the more the investment, the less the labor needed to operate the investment. Then the latest investment that uses up all available labor L is given by solving

$$L = b\int_{G'}^{G} g^{-n}\,dg \qquad \text{Full employment} \quad \text{p.7 eq.1}$$

$$= b\int_{g^{-1}(G')}^{g^{-1}(G)} g(\tau)^{-n}\frac{dg(\tau)}{d\tau}\,d\tau$$

with respect to G', which yields

$$G' = \left(G^{1-n}\frac{1-n}{b}L\right)^{\frac{1}{1-n}} \qquad \text{Oldest investment in use.}$$

Hence

$$x := a(G - G') \qquad \text{Output rate} \quad \text{p.7 eq.2}$$

$$= aG\left[1-\left(1-\frac{L}{\frac{b}{1-n}G^{1-n}}\right)\right]^{\frac{1}{1-n}} \qquad \text{p.7 eq.3}$$

The above calculation, which is easy by hand, can also be done with a computer algebra system if desired, as in Appendix A which uses MuPAD [7].

3 Solow's model

Let

$0 \le B_0$	Initial bound to improvement	p.28
$0 \le q < 1$	Amount of bound each innovation lowers	p.28
$k = 0, 1, \ldots$	Number of innovations	p.28
$0 < m$	Arrival rate of innovations	p.29
$0 \le P[\cdot] \le 1$	Probability.	

Solow's model bounds the improvement g^{-n} by $B_0\, q^k$, which may be written as

$$f(\tau) := B_0\, q^{k(\tau)} + bg(\tau)^{-n} \tag{1}$$

$$x(t) := \left\{ a[G(t) - G(t_0)] \;\middle|\; L = \int_{t_0}^{t} f(\tau) \frac{df(\tau)}{d\tau} d\tau \right\} \tag{2}$$

in which

$$\begin{aligned} P[k(\tau + d\tau) = k(\tau) + 1] &= m\, d\tau \qquad \text{p.29} \\ P[k(\tau + d\tau) = k(\tau)] &= 1 - m\, d\tau. \end{aligned} \tag{3}$$

A feature of this model is that $f(\tau)$ includes innovation, which is a function of exogenous innovation $k(\tau)$, in addition to Arrow's improvement term $b\, g^{-n}$.

The simulator and simulation runs found in the book deal with a partial model of growth in the sense that they all treat investment as being exogenous: see [9] p.44. Under the notation as introduced above it is easy to develop a complete simulation program. An implementation in R [5] is found in Appendix B.

4 Fluid approximation

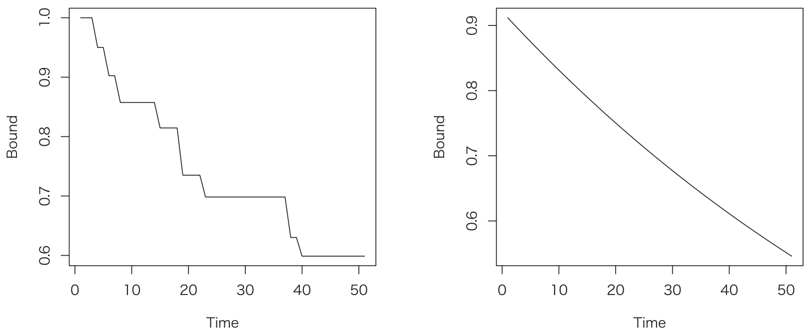
As can be seen from the simulation runs included in [9] it is not easy to understand the mechanism of Solow's model even with the help of a simulation program. Hoping to ameliorate the situation, the stochastic model is turned deterministic by adopting a fluid approximation. As is well known in queueing theory, fluid approximation is valid under *heavy traffic*, which is in our case when innovation takes place frequently. The approximation amounts to replacing (3) by

$$k(\tau + d\tau) = m\, d\tau.$$

The corresponding part in Appendix B is shown as a comment line in the program, in which $d\tau = 1$.

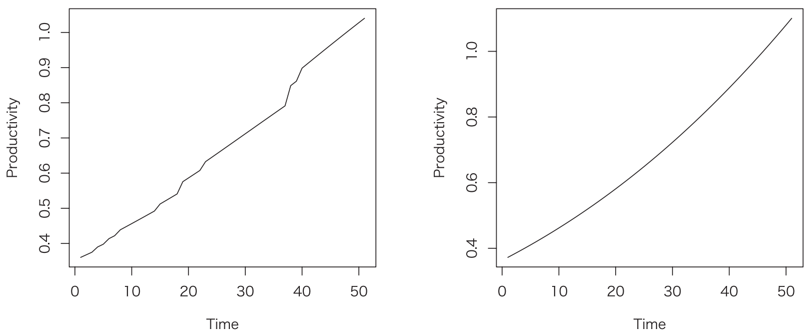
The program was used to produce both Poisson innovation and its fluid approximation shown in Figures 1 to 3. The first ten $d\tau$'s have been discarded in order to avoid the influence of the initial values. Note that in Figure 1 (a) the steps are clear while Figure 2 (a) is smoother. In Figure 3 the two graphs (a) and (b) are practically indistinguishable. Cumulative investments would seem even more identical since they are the graphs in Figure 3 integrated. Although these aspects clearly depend on parameters, it seems plausible to expect that the fluid approximation is fairly good when considered as an approximation to the growth even when it is a poor approximation to the bound of improvement.

The fluid approximation clarifies that, apart from the probabilistic complication and the nonexistence of closed form solutions, the overall structure of Solow's model is not that different from Arrow's: both have the form (2) with slightly different forms of $f(\cdot)$.



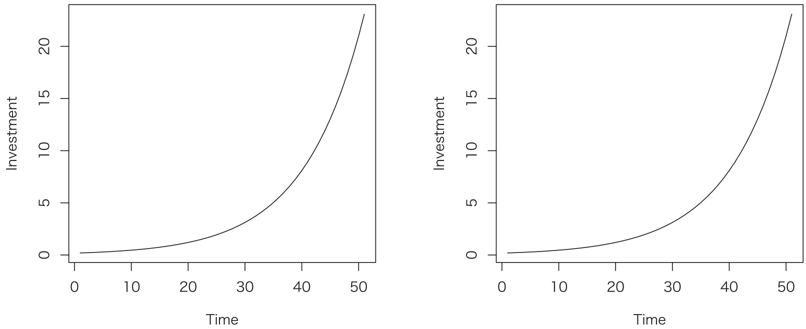
(a) Poisson innovation (b) Fluid innovation

Figure 1: Bound to improvement



(a) Poisson innovation (b) Fluid innovation

Figure 2: Productivity



(a) Poisson innovation (b) Fluid innovation

Figure 3: Investment

5 Too little is worse than nothing

The observation at the end of the previous section tempts one to modify $f(\tau)$ to incorporate some aspects of interest. A phenomenon of interest would be the existence of a threshold frequently observed in investment for technological progress: the effect of an investment is often worse than nothing when the amount allocated is insufficient to induce any progress. Such cases are of course common in real life: for instance, several meetings may be held producing no tangible result.

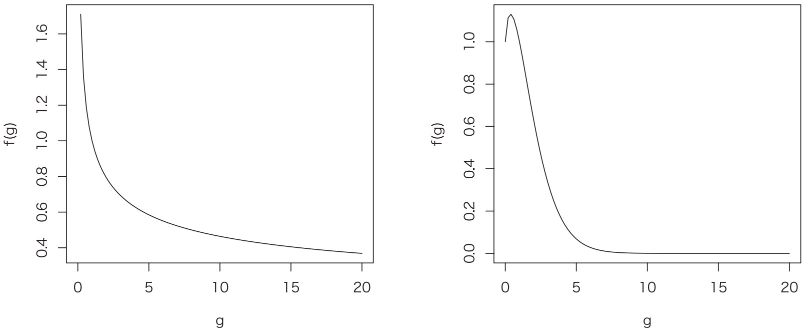
If the rate of improvement is made proportional to the investment rather than a constant n, in palce of Arrow's

$$f_a(g) = b\, g^{-n} \tag{4}$$

we have

$$f_c(g) = b\, g^{-n\, g} \tag{5}$$

which would look like in Figure 4 for $b = 1$ and $n = 1/3$. The expression (5)



(a) Arrow's improvement (b) Improvement with a barrier

Figure 4: Improvement functions

replaces the constant rate learning curve g^{-n} by g^{-ng} whose rate is now controlled by investment g, stating that investment accelerates learning.

Since the model does not permit a closed form solution, the difference in behavior between Arrow's and this model may be investigated by modifying the fluid approximation version of the simulator in Appendix B. For the same parameters as in the program, for instance, the productivity first exhibits a dip and then an increase, as in Figure 5.

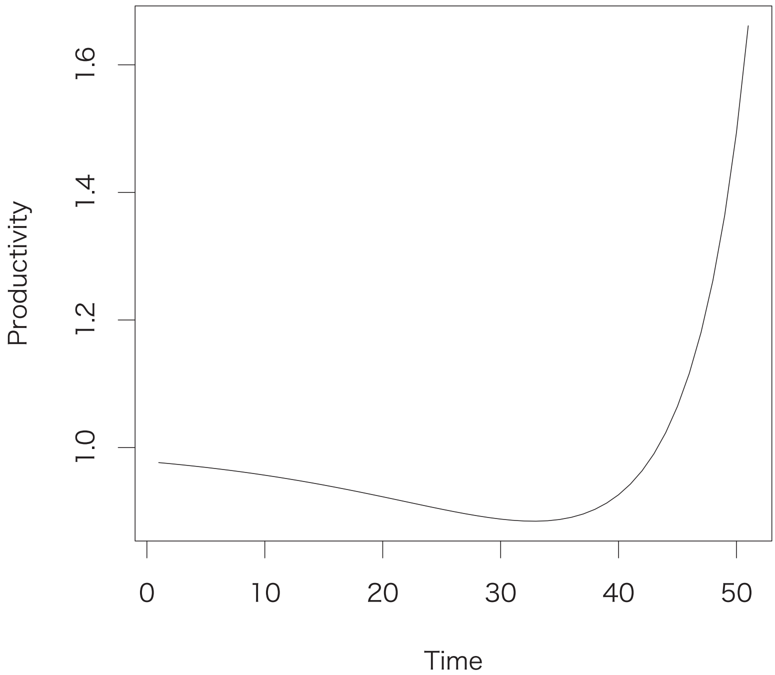


Figure 5: Productivity under investment barrier

6 Conclusion

The fluid approximation of Solow's model indicates that, when looked at mathematically, Arrow's and Solow's models differ in two major aspects:

1. While Arrow's is deterministic, Solow's is stochastic.
2. While Arrow's has a closed form solution, the fluid approximation to Solow's does not permit a general closed form solution.

Once the existence of a closed form solution is forsaken, an abundance of modifications becomes available to Arrow's model even without introducing stochastic behavior. One such possibility is the description that an investment may be too small to induce any progress.

References

[1] K. J. Arrow. The economic implications of learning by doing. *Review of Economic Studies*, 14 (28) :155-73, 1962.

[2] K. B. Luintel and M. Khan. Heterogeneous ideas production and endogenous growth: An empirical investigation. 2005. http://www.core.ucl.ac.be/news/seminars/luintel.pdf.

[3] The New School for Social Research. *Growth Theory*. http://cepa.newschool.edu/het/essays/growth/growthcont.htm.

[4] The New School for Social Research. *The History of Economic Thoughts*. http://cepa.newschool.edu/het/.

[5] R Development Core Team. *R: A language and environment for statistical computing*. R Foundation for Statistical Computing, Vienna, Austria, 2005. ISBN 3-900051-07-0, http://www.R-project.org/.

[6] P. M. Romer. Economic growth. In D. R. Henderson, editor, *The Concise Encyclopedia of Economics*. Liberty Fund, 2007 (forthcoming). http://www.stanford.edu/~promer/EconomicGrowth.pdf.

[7] SciFace Software. *MuPAD*, 2004. http://www.mupad.de/.

[8] R. M. Solow. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70 (1) :65-94, February 1956.
[9] R. M. Solow. *Learning from 'Learning by Doing' –Lessons for Economic Growth*. Stanford University Press, 1997.

Appendixes

A Derivation with MuPAD

Figure 6 is a sample session to derive Arrow's model in closed form using MuPAD.

```
l := int(g^-n,g=G_dash..G,Continuous)
```

$$\frac{G\_dash^{1-n}}{n-1}-\frac{G^{1-n}}{n-1}$$

```
eq := L = subs(l,G_dash^(1-n)=H)
```

$$L=\frac{H}{n-1}-\frac{G^{1-n}}{n-1}$$

```
h := op(solve(eq,H))
```

$$(n-1)\cdot\left(L+\frac{G^{1-n}}{n-1}\right)$$

```
a*(G - h^(1/(1-n)))
```

$$a\cdot\left(G-\frac{1}{((n-1)\cdot(L+\frac{G^{1-n}}{n-1}))^{\frac{1}{n-1}}}\right)$$

```
x := simplify(%)
```

$$a\cdot\left(G-\frac{1}{\left(\frac{G-G^{n}\cdot L+G^{n}\cdot L\cdot n}{G^{n}}\right)^{\frac{1}{n-1}}}\right)$$

Figure 6: Derivation of Arrow's model

B Simulation program in R

This is a simulator for Solow's model in R.

```
# Innovation and improvement
# File: growth.R
# 2005.11 yoneda

# R. M. Solow; Learning from Learning by Doing;
# Stanford 1997
# Assume dt = 1
simlen <- 60    # Simulation length
labor <- 1      # Total available labor
q <- 0.95       # Innovation ratio; p.45
m <- 0.2        # Innovation arrival rate; p.45
                # Innovations are exogenous.
n <- 1/3        # Improvement rate; p.4
b <- 1          # Importance of improvement
                # against innovation; p.44
B <- array(dim=simlen)
    # Present bound to innovation; p.25
B[1] <- 1       # Initial bound to improvement; p.44
a <- 1          # Output factor to
                # culumative investment; p.44
r <- 0.1        # Investment factor to ouotput;
                # not in the book.
g <- array(dim=simlen)
                # g[t] = Investment at time t
g[1] <- 1       # Initial investment
    # G and hence g are exogenous in p.44
    # but endogenous here.
G <- array(dim=simlen)
                # G[t] = Cumulative gross investment
                # up to time t
G[1] <- 1
f <- array (dim=simlen)
                # f[t] = Technology at time t
```

```
f[1] <- 1
k <- function(m) rpois(1,m) # Innovation arrival
# k <- function(m) m # Fluid approximation
tech <- function(t)     # Technology, high tech = low value
    B[t] + b*g[t-1]^(-n)
    # b*g[t-1]^(-n)
    # b*g[t-1]^(-n*g[t-1])
t0 <- function(t) {     # Time for oldest investment in use
    tFr <- t-1; l <- 0
    while ( l <= labor && 2 <= tFr ) { # int f df/dt dt
        l <- l + f[tFr]*(f[tFr]-f[tFr-1])
        tFr <- tFr - 1 }
    if (l <= labor) tFr + 1 else 1 }
    # Overuses labor but not by much.
xf <- function(t)       # Aggregate output at time t
    a*sum(g[t0(t):t])
gf <- function(t)       # Investment
    if (2 <= t) r * xf(t-1) else g[1]
p <- array(dim=simlen)  # Productivity
p[1] <- 1
# Simulate
for (t in 2:simlen) {
    B[t] <- B[t-1] * q^k(m)
    f[t] <- tech(t); g[t] <- gf(t); G[t] <- G [t-1] + g[t];
    p[t] <- a/f[t] }
# Plot
tFrom <- 10
plotFr <- function(a="p",fr=10) {
    if (a == "B") { y <- B; yl <- "Bound" }
    if (a == "g") { y <- g; yl <- "Investment" }
    if (a == "G") { y <- G; yl <- "Cumulative investment" }
    if (a == "p") { y <- p; yl <- "Productivity" }
    plot(y[tFrom:simlen],
        type="l",lty="solid",xlab="Time",ylab=yl) }
```