The Monty Hall Dilemma with Joint Sufficient Statistics☆

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Abstract:

The Monty Hall's TV show provides a host-guest game, in which a guest, a lady makes 2*-stage decisions to win a prize by picking out one of* 3 *choices that a host has offered. After the guest makes a choice for the first stage, the host reveals another choice does not give her the prize. If the guest believes equally likely events to her prior knowledge or indifferent probabilities of winning the prize, then she will fall into a dilemma between staying with the first choice and switching it to the other left. However, to avoid the dilemma, the guest may utilize a sample data, the past TV game shows: That is, she is able to have quasi experiences from the past shows in learning by her doing of either staying or switching to win the prize. In this paper, we would like to study how such a guest, who is supposed to be a Bayesian, revises her prior knowledge with those data one after another. So as to describe her knowledge, we employ a Dirichlet distribution.*

Keywords: Monty Hall; Dilemma; Equally likely; Prior knowledge; Quasi experience; Learn by doing; Bayesian; Dirichlet distribution.

☆ This study is partly supported by the Jutenka Senmon Yiinkai (#.994001) of the Center Reserch Institute Fukuoka University, Fukuoka, Japan.

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1. INTRODUCTION

Say it is known that there are 3 opaque boxes A, B, and C, each of which contains only one chip among 2 gray and 1 red chips, and that the chips are identical in size, weight, and so on, but in color. First of all, a host asks a guest, who is supposed to be a lady for convenience' sake, a game in which he gives her a winning prize such as a diamond ring if she picks out the box that contains the red chip. If not, then he gives her nothing. Secondly, the guest picks up a box whichever she might like according to her first hunch or prior knowledge. Thirdly, the host, who knows whereabouts of the red chip, reveals in front of her that another box contains a gray chip. Fourthly, the guest is allowed to make another decision to stick with the first hunch or switch it to the other box left. Finally, the host shows which box has contained the red one.

The above host-guest game, in which a guest makes 2-stage decisions to obtain a prize, comes from a TV game show, *Let's Make a Deal*. Monty Hall, the show's host has seen over 4, 500 programs that a contestant as the guest often gets into a dilemma between staying with an initial choice and switching it to the other remaining after his revelation. Statistically speaking, the dilemma is caused by the reason why the guest intuitively expects in her mind that the events of winning the prize are *equally likely,* so that each choice has the equal probability of winning it as a ratio of the unity to the total number T of choices or $\frac{1}{T}$: The probability of the win on the first stage is $\frac{1}{3}$; but that becomes $\frac{1}{2}$ on the second stage since the total number of choices is reduced by the revelation from $T=3$ to $T=2$. Hence, she suffers from the dilemma because of indifferent probabilities between 2 choices still left after one candidate of the prize has gone.

Vos Savant (1990a) astonishes those, who believe the events of winning a prize on the Monty Hall's TV game show should be equally likely, by complaining that the probability of winning the prize by staying with an original choice should not be $\frac{1}{2}$ derived from $\frac{1}{T}$ when $T=2$ but $\frac{1}{3}$ whereas the probability by switching the original choice to the other left should be $\frac{2}{3}$ instead of $\frac{1}{2}$ given by $(1-\frac{1}{T})$ when $T=2$. It does not seem successful that vos Savant (1990b) proves her $\frac{2}{3}$ solution to convert those believers of equally likely events;[1] However, a couple of Bayesian analyses to a 31 years old (at that time) problem of "the 3-prisoner problem" have come to stand by her because it has a similar structure to the TV game show. For example, Morgan, Chaganty, Dahila, and Doviak (1991) improve Mosteller (1965)'s sample space analysis to solve it as an unconditional probability problem and support vos Savant's scenario for the $\frac{2}{3}$ solution; In addition, Gardner (1992) mentions that he might be a pioneer (in a 1959 column) to show the $\frac{2}{3}$ solution by the Bayes' theorem. They make use of the host's revelation strategy as a key to apply the theorem. To briefly see this, assume without losing any generality that the guest chose Box A with her original hunch of equally likely events and then the host would reveal Box B as a gray chip. Let A, B, and C denote an event that Box A, B, and C contains the red chip, respec-

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[1] Her proof looks like as if one substituted $T = 3$ into $(1-1/T)$; but, Selvin (1975) has already shown a mathematical formula to compute this kind of probability, giving D. L. Ferguson credit for the formula, which can be written as $T/\{(T-1)(T+1)\}$ after the host's revelation or at $T=2$ and so it provides that $2/\{(2-1)(2+1)\} = 2/3$.

tively. Then, she believes indifferent probabilities of the red chip in boxes or $P_r(A) = P_r(B) = P_r(C) = \frac{1}{3}$. Let $\hat{b}$ be the host's strategy to reveal Box B as a gray chip, and let $\hat{c}$ be the strategy for Box C. One may think it sensible as discussed in Lindley (1971) that if Box A contains the red chip, then he will reveal Box B or C with conditional probabilites of $P_r(\hat{b}|A) = P_r(\hat{c}|A) = \frac{1}{2}$; if Box B contains it, he must reveal Box C with $P_r(\hat{b}|B)=0$ and $P_r(\hat{c}|B)=1$; and if Box C has the red one, he reveals Box B with $P_r(\hat{b}|C)=1$ and $P_r(\hat{c}|C)=0$. So, the Bayes' theorem is able to produce

$$P_r(C|\hat{b}) \equiv \frac{P_r(\hat{b}|C) \times P_r(C)}{P_r(\hat{b})}$$

$$= \frac{(1)(1/3)}{(1/2)(1/3)+(0)(1/3)+(1)(1/3)} = \frac{2}{3}$$

where $P_r(\hat{b}) \equiv P_r(\hat{b}|A) \times P_r(A) + P_r(\hat{b}|B) \times P_r(B) + P_r(\hat{b}|C) \times P_r(C)$. Accordingly, it seems to be widely accepted these days that vos Savant's complaint or her $\frac{2}{3}$ solution is correct due to the sensible strategies for the host's revelation.[2]

However, it is notorious that the Gardner's Bayesian analysis with the host's revelation strategy often derives a counterintuitive solution. For instance, say it is known to a guest as her prior knowledge that the probabilities of obtaining the red chip are $P_r(A) = \frac{1}{2}$, $P_r(B) = \frac{1}{8}$, and $P_r(C) = \frac{3}{8}$. Other things being equal to the previous assumption, the analysis gives $P_r(C|\hat{b}) = (1)(3/8)/\{(1/2)(1/2)+(0)(1/8)+(1)(3/8)$ $=3/5$ and so $P_r(A|\hat{b}) = 1-3/5 = 2/5$. It is interesting to see firstly that $P_r(A|\hat{b}) = \frac{2}{5}$ does not remain same as $P_r(A) = \frac{1}{2}$ even if not only Gardner but his advocates insist that it should remain same because of the reason why the host's revelation provides *no* new information about the red chip's whereabouts; secondly that $P_r(A|\hat{b})=\frac{2}{5}$ is less than $P_r(A)=\frac{1}{2}$ regardless of the reason why Box B is no longer a candidate for the red chip so that one might expect that at least $P_r(A|\hat{b})\geq P_r(A)$ should hold; and thirdly that the guest has to change her mind from Box A to C since $P_r(A|\hat{b})=\frac{2}{5}$ is less than $P_r(C|\hat{b})=\frac{3}{5}$ although she knows that Box A has the highest probability among them or $P_r(A)=\frac{1}{2}>P_r(C)=\frac{3}{8}$. Since it runs counter to one's intuition, one may wonder how many people actually use this sort of analysis:[3] Shimojo and Ichikawa (1989) have 95 examinees to the 3-prisoner problem, and report with $P_r(A)=P_r(B)=\frac{1}{4}$ and $P_r(C)=\frac{1}{2}$ that only 2% reach a Gardner's solution of $P_r(A|\hat{b})=\frac{1}{5}$; Regarding the Monty Hall's game, Granberg and Brown (1995) report with 228 examinees (but without any probability value) that just 13% switch in the first encounter to the game, and also report with 114

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[2] Hoffman (1998) writes an episode about Erdös, one of the greatest mathematicians in the last century, as an example that "his intuition was not always perfect." When Vázsonyi tells Erdös about the dilemma that the correct answer after the revelation is to switch, he replies 'No, that is impossible. It should make no difference.' Vázsonyi employs a decision tree for a Bayesian analysis, but it does not convince Erdös at all. One may think that nothing is wrong with Erdös' intuition because the 2/3 Bayes' solution could have been correct if the revelation strategy $\hat{b}$, which is impossible to be observed by the guest, were known and given to her.

[3] One may wonder whether or not the Gardner's Bayesian analysis is useful. Fujimoto (2001) shows that the analysis might be useless for a rational guest who minimizes a square-error loss function to take expectations for the red chip's whereabouts since any Gardner's solution fails to minimize the loss function. Because the host's strategy $\hat{b}$ is never to be observable (nor even to be measurable) by a guest as stated in Footnote 2, it is redundant for the rational guest to expect $\hat{b}$ by a gray chip to expect the red chip's whereabouts, so that the Gardner's solution can have a larger variance than an optimal solution given by the loss function has.

examinees that across 50 trials, the average of switching goes up from 12% to 55%.

Therefore, only few people seem to utilize the Gardner's Bayesian analysis because it hardly makes an appeal to one's intuition, and no one seems to make use of it and learn the guest's always-switch strategy, which must be suggested by a Gardner's solution such as vos Savant's $\frac{2}{3}$ one, since it is impossible for one to observe the host's strategy as mentioned in Footnote 2. Moreover, even though advocates of the Gardner's Bayesian analysis usually blame those poor results or performances not on the Gardner's analysis but on student examinees as if they were not rational at all times, the analysis itself seems to be irrational as discussed in Footnote 3. Furthermore, it seems to be difficult tasks for such a classical Bayesian analysis, in which any prior knowledge to an experiment is treated as an unknown constant, to revise the knowledge by a sample and take into account its weight information or a sample size n as Anand (1991) claims.

Our main purpose is to study how a guest, who takes into account a sample like past TV shows, revises her prior knowledge by the sample one after another. In order to achieve the purpose, we assume the guest is a modern Bayesian who assigns a Dirichlet distribution to her prior knowledge, and proceed with this paper as follows: In Section 2, we consider the host-guest game as a so-called *no data* problem in Berger (1985) where the host has already revealed a gray chip but not yet shown the final result or datum of the red chip. We have a 3-point or multi-Bernoulli distribution because there are only 3 possible outcomes of the red chip's whereabouts in the beginning of the game. It will be shown that after the gray chip is revealed by the host, the guest has a conditional mathematical expectation given that gray chip as her optimal decision so as to estimate the red chip's whereabouts; In Section 3, we consider the game as a so-called *Bayesian statistics* problem in Hogg and Craig (1995) where any prior knowledge to an experiment that a statistician has is treated as a random variable because the 3-point distribution depends upon 2 population parameters of experimental values of random variables. We assume that the guest is rational enough to minimize a Bayes' risk, an expected square-error loss function given joint sufficient statistics for 2 parameters in obtaining a Bayes' solution or an optimal decision function. It will be shown that the Bayes' solution is nothing but the mathematical expectations or the mean of optimal decision for the no data problem studied in Section 2, and it is a weighted average of a maximum likelihood estimate and the mean of a prior probability density function of 2 parameters. One may expect the Bayes' solution converge to the maximum likelihood estimate as the number of trials or the sample size n goes up, which seems to verify the phenomenon in Granberg and Brown that across 50 trials, the average of switching increases from 12% to 55%; In Section 4, we provide a conclusion.

2. A NO DATA PROBLEM

To begin with, let us put the host-guest-box-chip game into the following processes in order to find out *induced probabilities* for the game.

Process 0: Preparation] As a host, a gentleman prepares 3 opaque boxes,

A, B, and C, each of which contains only one chip among 2 gray and

1 red chips. All 3 chips are identical in shape, size, and so forth but in color. Let F be a function for a random variable such that $F(\chi)=0$ if χ is a gray chip G and $F(\chi)=1$ if χ is the red chip R.

Process 1: Host's first move] He asks a guest, who is supposed to be a lady for convenience' sake, a game where if she picks out the box which contains the red chip, then he will give her a nice present such as a diamond ring; but if not, he gives her nothing.

Process 2: Guest's first move] The guest chooses a box, say Box A whichever she may like according to her first hunch or prior knowledge. In other words, the choice is made based upon the probabilities $P_r(A)$, $P_r(B)$, and $P_r(C)$ in which A, B, and C is the event that each box contains the red chip. Let A and a be a random variable for Box A and its particular realization, then one has $a=0$ if Box A contains a gray chip or $A(G)=0$ whereas $a=1$ if Box A contains the red chip or $A(R)=1$.

Process 3: Host's second move] The host reveals that another box, say Box B contains one of 2 gray chips. Let B and b denote a random variable for Box B and its particular realization, so that one has $b=0$ if Box B contains a gray chip or $B(\mathrm{G})=0$ and $b=1$ if Box B has the red chip or $B(R)=1$. Therefore, she observes the realization or datum of $b=0$ by the revelation.

Process 4: Guest's second move] The guest makes another decision of whether to stick with her original choice Box A or to change it into the other Box C by thinking of $P_r(A|b=0)$ and $P_r(C|b=0)$, the conditional probabilities of the events given the gray chip revealed by the host in Process 3.

Process 5: Host's third move] The final result comes out: The host shows her if Box A or C has contained the red chip, that is, if she can get the diamond ring or not. This process is missing in both the 3-prisoner problem and the Monty Hall's dilemma game. According to Berger (1985), those games still belong to a *no data* problem since without any realization or datum a, she has not yet learnt anything by her doing of staying with Box A or switching it to win the ring. This process will be studied in the next section.

Now, we can define a space for the random variables A and B as a *discrete* set $\triangle$ of 3 pairs of $(a,\ b)$'s with their particular realizations:

$$\triangle \equiv \{(a,\ b) \mid (0,\ 0),\ (1,\ 0),\ (0,\ 1)\},$$

which reflects that there are 3 possible outcomes of the events that each box contains the red chip R: That is, (1, 0) represents for the event A that Box A contains R; (0, 1) stands for the one B that Box B contains R; and so that (0, 0) means the one C that Box C has R since it must happen if each of Box A and B contains a gray chip or if $a=b=0$. Let θ and ρ denote a population parameter for $P_r(A)$ and $P_r(B)$, respectively. Then, $P_r(C)$ is expressed as $(1-\theta-\rho)$ because $P_r(A)+P_r(B)+P_r(C)=1$. Therefore, we can see that A and B jointly have a 3-point or multi-Bernoulli distribution whose joint probability mass function $f(a,\ b)$ can be written as

$$f(a,\ b)\begin{cases} \dfrac{1}{a!b!(1-a-b)!}\theta^a\rho^b(1-\theta-\rho)^{1-a-b} & \text{if}(a,\ b) \in \triangle, \\ 0 & \text{elsewhere}, \end{cases} \tag{1}$$

in which the exclamation mark ! stands for the factorial; $0!=1!=1$. Let $E\{\ \}$ be an operator for the mathematical expectations, then we have

$$E\{A\} \equiv \sum_{a=0}^{1} \sum_{b=0}^{1-a} af(a,\ b)$$

$$= \sum_{a=0}^{1} \frac{a\theta^a}{a!(1-a)!} \sum_{b=0}^{1-a} \frac{(1-a)!}{b!(1-a-b)!} \rho^b (1-\theta-\rho)^{1-a-b}$$

$$= \sum_{a=0}^{1} \frac{a\theta^a}{a!(1-a)!} (1-\theta)^{1-a} = \theta,$$

$$E\{B\} \equiv \sum_{b=0}^{1} \sum_{a=0}^{1-b} bf(a,\ b) = \rho$$

in a similar manner. So, the probabilities on the 3 points can be expressed by

$$f(1,\ 0) = P_r(A) = \theta = E\{A\}, \tag{2}$$

$$f(0,\ 1) = P_r(B) = \rho = E\{B\}, \tag{3}$$

$$f(0,\ 0) = P_r(C) = 1 - \theta - \rho. \tag{4}$$

Based on the probabilities (assuming they are unknown constants: $0 < \theta < 1$; $0 < \rho < 1$; and $0 < 1 - \theta - \rho < 1$), which stand for the guest's first hunch or prior knowledge, she is supposed to make her initial choice of Box A in Process 2].

Since a marginal function of *Equation* (1) for B, say $f_b(b)$ is calculated as

$$f_b(b) \equiv \sum_{a=0}^{1-b} f(a,\ b) = \frac{\rho^b}{b!(1-b)!} \sum_{a=0}^{1-b} \frac{(1-b)!}{a!(1-a-b)!} \theta^a (1-\theta-\rho)^{1-a-b}$$

$$= \frac{1}{b!(1-b)!} \rho^b (1-\rho)^{1-b} \tag{5}$$

if $b=0,\ 1$ and $f_b(b)=0$ elsewhere, a conditional probability mass function of A given a particular realization $B=b$ denoted by $f(a|b)$ becomes

$$f(a|b) \equiv \frac{f(a,\ b)}{f_b(b)} = \frac{(1-b)!}{a!(1-a-b)!}\ \frac{\theta^a(1-\theta-\rho)^{1-a-b}}{(1-\rho)^{1-b-a+a}}$$

$$= \frac{(1-b)!}{a!(1-a-b)!}\left(\frac{\theta}{1-\rho}\right)^a\left(\frac{1-\theta-\rho}{1-\rho}\right)^{1-a-b} \tag{6}$$

if $a=0,\ 1-b$ and $f(a|b)=0$ elsewhere. Let both d and u be dummy variables such that $d \equiv a-1$ and $u \equiv 1-b$, and *Equation* (6) provides us with $E\{A|b\}$, conditional expectations of A given a realization $B=b$ as

$$E\{A|b\} \equiv \sum_{a=0}^{1-b} af(a|b) = \sum_{a=0}^{u} \frac{u!}{d!(u-a)!}\left(\frac{\theta}{1-\rho}\right)^a\left(\frac{1-\theta-\rho}{1-\rho}\right)^{u-a}$$

$$= \frac{\theta}{1-\rho}\sum_{d=0}^{u-1}\frac{u(u-1)!}{d!(u-d-1)!}\left(\frac{\theta}{1-\rho}\right)^d\left(1-\frac{\theta}{1-\rho}\right)^{u-d-1}$$

$$= u\frac{\theta}{1-\rho}\left(\frac{\theta}{1-\rho}+1-\frac{\theta}{1-\rho}\right)^{u-1}$$

$$= (1-b)\frac{\theta}{1-\rho} \tag{7}$$

if $b=0,\ 1$. After the guest observes Box B has contained a gray chip G or $b=0$, not only *Equation* (6) but (7) gives her the conditional probabilities given $b=0$:

$$P_r(A|b=0) = f(a=1|b=0) = \frac{\theta}{1-\rho} = E\{A|b=0\} \tag{8}$$

$$P_r(C|b=0) = f(a=0|b=0) = 1-\frac{\theta}{1-\rho}; \tag{9}$$

$$P_r(B|b=0) = 1-P_r(A|b=0)-P_r(C|b=0) = 0. \tag{10}$$

Consequently, we have the following theorems that can save one's intuition.

Theorem 1: *After the host shows Box B has contained a gray chip, not only $P_r(A)$ but also $P_r(C)$ increases as long as $P_r(A)$, $P_r(B)$, and $P_r(C)$ are positive.*

Proof: Subtract *Equation* (2) from *Equation* (8), and we have

$$P_r(A|b=0) - P_r(A) = \frac{\theta}{1-\rho} - \theta = \frac{\theta\rho}{1-\rho} > 0.$$

It holds if $P_r(A) \equiv \theta > 0$ and $P_r(B) \equiv \rho > 0$. On the other hand, subtracting *Equation* (4) from *Equation* (9) yields

$$P_r(C|b=0) - P_r(C) = 1 - \frac{\theta}{1-\rho} - (1-\theta-\rho) = \frac{\rho(1-\theta-\rho)}{1-\rho} > 0.$$

It is always true if $P_r(B) \equiv \rho > 0$ and $P_r(C) \equiv 1 - \theta - \rho > 0$. ***Q. E. D.***

Theorem 2: *After the host reveals Box B has had a gray chip, the guest again falls into a dilemma if and only if $P_r(A) = P_r(C)$.*

Proof: ⇒) Set *Equation* (8) equal to $\frac{1}{2}$ for the dilemma, and one has

$$\theta = 1 - \theta - \rho \tag{11}$$

whose left hand side is nothing but $P_r(A)$ and right hand side is $P_r(C)$ itself. ⇐) Suppose that $P_r(A) = P_r(C)$ in the beginning of the game. Its expression in the population is *Equation* (11). Recall that $0 < \rho < 1$, then $0 < 1 - \rho < 1$. Divide both sides of *Equation* (11) by $(1-\rho)$, and we have

$$P_r(A|b=0) = \frac{\theta}{1-\rho} = 1 - \frac{\theta}{1-\rho} = P_r(C|b=0).$$

This indifference makes her get into the dilemma. ***Q. E. D.***

Theorem 3: *After the host reveals Box B has had a gray chip, a large-small relation between $P_r(A)$ and $P_r(C)$ keeps order: That is, if $P_r(A) > P_r(C)$, then $P_r(A|b=0) > P_r(C|b=0)$; but if $P_r(A) < P_r(C)$, $P_r(A|b=0) < P_r(C|b=0)$.*

Proof: Suppose that $P_r(A) > P_r(C)$ or $\theta > 1 - \theta - \rho$. Dividing both sides by a positive $(1-\rho)$ gives us that

$$P_r(A|b=0) = \frac{\theta}{1-\rho} > 1 - \frac{\theta}{1-\rho} = P_r(C|b=0).$$

In a similar way, we have

$$P_r(A|b=0) = \frac{\theta}{1-\rho} < 1 - \frac{\theta}{1-\rho} = P_r(C|b=0)$$

if $P_r(A) < P_r(C)$ or $\theta < 1 - \theta - \rho$. ***Q. E. D.***

Theorem 1 verifies an intuition that Box B is no longer likely for the red chip's whereabouts so that the other 2 likelihood or the probabilities should increase. Theorem 2 verifies an Erdös' intuition discussed in Footnote 2 that if the events that each box contains the red chip are equally likely, then the host's revelation of the gray chip should make no difference between the other 2 probabilities. Theorem 3 verifies an intuition that the host's revelation is not a hint for the red chip's whereabouts and the guest has not yet had a particular realization to revise her first hunch or prior knowledge so that on the second stage she cannot help acting based on what she has believed as her prior knowledge on the first stage. Let Max { , } denote an operator for the maximum value which takes the bigger one than the other and let Min { , } be for the minimum value, then one may think of 4 strategies that she can employ for the belief on the first stage.

Strategy 1: Equally likely; $\theta = \rho = 1 - \theta - \rho = \frac{1}{3}$.]

It means that the guest chooses Box A because there is no difference among boxes or she believes that each box equally likely contains the red chip.

Strategy 2: Most favorite; $\theta \geq \mathrm{Max}\{\rho,\ 1-\theta-\rho\}$.]

The guest chooses Box A frankly because she makes a guess somehow that this box has the highest possibility to contain the red chip or she likes this box the most. Thus, she does not switch this box in Process 4].

Strategy 3: Second favorite; i) $1-\theta-\rho \leq \theta < \rho$ or ii) $\rho \leq \theta < 1-\theta-\rho$.]

This means that the guest chooses the second favorite box for the first time. Hence, she will switch from Box A to C in Process 4] only if the most favorite one in her mind is not ridded by the host in Process 3].

Strategy 4: Least favorite; $\theta \leq \mathrm{Min}\{\rho,\ 1-\theta-\rho\}$.]

It is kind of perverse: The guest selects Box A in Process 2] even if she is somehow sure that either of the other 2 boxes contains the red chip. She always switches Box A to C in Process 4] no matter which box the host might reveal as a gray chip in Process 3].

Hence, she had better make some difference among 3 boxes (especially, between Box A and C, and at least shortly) before the host reveals Box B as a gray chip. Otherwise, she would suffer from the dilemma in Process 4].

3. A BAYESIAN ANALYSIS

In this section, we repeatedly analyze the host-guest game with Process 5], in which the host finally shows the guest whether or not she could successfully pick out the box that has contained the red chip to win a nice present. Because after this process, one has particular realizations

a and b or an element (a, b) in the discrete set $\triangle \equiv \{(a, b) | (0, 0), (1, 0), (0, 1)\}$, the word "repeatedly" means that the guest is assumed to make use of a sample with size n and have quasi (or actual) experiences "n times" from the sample in learning by doing of the staying-switching decision so that she can "repeatedly" revise her prior knowledge one after another. Recall that $P_r(A) \equiv \theta$ and $P_r(B) \equiv \rho$. To describe her prior knowledge against the events A, B, and C that each box contains the red chip or the population parameters θ and ρ, we assign a Dirichlet distribution to her knowledge with positive hyper parameters α, β, and γ, reflecting degrees of her belief on the events A, B, and C, respectively.[4] Let D be a domain set of

$$D \equiv \{(\theta, \rho) | 0 < \theta \leq 1 - \rho < 1\},$$

then its probability density function π is expressed as

$$\pi(\theta, \rho) = \begin{cases} \dfrac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \theta^{\alpha-1} \rho^{\beta-1} (1-\theta-\rho)^{\gamma-1} & \text{if } (\theta, \rho) \in D \\ 0 & \text{elsewhere,} \end{cases} \tag{12}$$

in which $\Gamma(\ )$ is a γ function well defined as $\Gamma(m) \equiv \int_0^\infty z^{m-1} e^{-z} dz$.

Let Q and R be a random variable whose particular realization is θ and ρ, respectively. Since we have the mean of Q as

[4] This assignment has 4 advantages: Firstly, a parametric analysis is more sensitive than a nonparametric one; Secondly, we can describe a lot of distributions on a region of (0, 1) because there are infinite combinations of α, β, and γ; Thirdly, we can coherently transmit *personal* or *subjective* belief on the events A, B, and C to probabilities as well as moments (if any) through sufficient statistics for θ and ρ; Last of all, it is tractable for us to compute a posterior distribution owing to a same kernel such as $\theta\rho(1-\theta-\rho)$ in *Equation* (1) and (12).

$$E\{Q\} \equiv \int_0^1 \int_0^{1-\rho} \theta\pi(\theta, \rho)\, d\theta d\rho = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \frac{\Gamma(\alpha+1)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+1+\beta+\gamma)},$$

the probability of the event A that Box A contains the red chip is given by

$$P_r(A) = E\{E\{A\}\} = E\{Q\} = \frac{\alpha}{\alpha+\beta+\gamma}. \qquad (13)$$

In a same manner, we have

$$P_r(B) = E\{E\{B\}\} = E\{R\} = \frac{\beta}{\alpha+\beta+\gamma}, \qquad (14)$$

$$P_r(C) = 1 - E\{Q\} - E\{R\} = \frac{\gamma}{\alpha+\beta+\gamma}. \qquad (15)$$

They correspond to *Equation* (2) through (4) in Section 2.

Let $\underline{S} \equiv \{(A_1, B_1), (A_2, B_2), \cdots, (A_n, B_n)\}$ denote a random sample with size n from *Equation* (1). Let $X \equiv A_1 + A_2 + \cdots + A_n$ and $Y \equiv B_1 + B_2 + \cdots + B_n$ be a sum of each A_i and each B_i, respectively for a running index $i = 1, 2, \cdots, n$. Besides, let $\underline{s}$, x, and y be their particular realizations: That is, $\underline{s} \equiv \{(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)\}$; $x \equiv a_1 + a_2 + \cdots + a_n$; and $y \equiv b_1 + b_2 + \cdots + b_n$. Moreover, define 2 nonnegative functions k_1 and k_2:

$$k_1 \equiv \theta^x \rho^y (1-\theta-\rho)^{n-x-y} > 0; \qquad (16)$$

$$k_2 \equiv 1 > 0.$$

Furthermore, let $\Pi[f]$ be a function for a product of f's from *Equation* (1), then the product of probabilities of the sample $\underline{s}$ becomes

$$\Pi[f(a_i, b_i)] \equiv \theta^{a_1}\rho^{b_1}(1-\theta-\rho)^{1-a_1-b_1} \cdots \theta^{a_n}\rho^{b_n}(1-\theta-\rho)^{1-a_n-b_n}$$

$$= \theta^x \rho^y (1-\theta-\rho)^{n-x-y} = k_1 \times k_2 \qquad (17)$$

because each of both a_i and b_i takes either 0 or 1 for the index $i=1, 2, \cdots, n$, and makes all factorials 1: $0! = 1! = 1$. Since we can see that *Equation*

(17) is factorized by nonnegative functions k_1 and k_2, the sums X and Y are *joint sufficient statistics* for θ and ρ.[5] We can also see from the kernel of k_1 that the sums X and Y have a trinomial distribution with a mass function $g(x, y)$ of

$$g(x,\ y)=\begin{cases}\dfrac{n!}{x!y!(n-x-y)!}\,\theta^x\rho^y(1-\theta-\rho)^{n-x-y} & \text{if}(x,\ y)\in N_i\,, \\ 0 & \text{elsewhere,}\end{cases} \tag{18}$$

where N_i is a set of a pair of nonnegative integers such that x and y take $0, 1, \cdots, x+y \leq n$ at most. By the definition of a conditional probability we have

$$\pi(\theta,\ \rho\,|\,\underline{s})\equiv\frac{\pi(\theta,\ \rho)k_1}{\int_0^1\int_0^{1-\rho}\pi(\theta,\ \rho)k_1\,d\theta d\rho}$$

$$=\frac{\pi(\theta,\ \rho)g(x,\ y)}{\int_0^1\int_0^{1-\rho}\pi(\theta,\ \rho)g(x,\ y)\,d\theta d\rho}\equiv\pi(\theta,\ \rho\,|\,x,\ y),$$

which tells us that a posterior density function given joint sufficient statistics, $\pi(\theta, \rho\,|x, y)$ is similar to the one given the sample, $\pi(\theta, \rho\,|\underline{s})$ and that the function is proportionally varying (denoted by $\propto$) to the kernel in the numerators like

$$\begin{aligned}\pi(\theta,\ \rho\,|\,\underline{s}) &\propto \pi(\theta,\ \rho)g(x,\ y)\\ &\propto \theta^{\alpha-1}\rho^{\beta-1}(1-\theta-\rho)^{\gamma-1}\,\theta^x\rho^y(1-\theta-\rho)^{n-x-y}\\ &= \theta^{\alpha+x-1}\rho^{\beta+y-1}(1-\theta-\rho)^{\gamma+n-x-y-1}.\end{aligned}$$

Let ξ be a constant coefficient such that

$$\xi\equiv\frac{\Gamma(\alpha+\beta+\gamma+n)}{\Gamma(\alpha+x)\Gamma(\beta+y)\Gamma(\gamma+n-x-y)},$$

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[5] See, for example, Hogg and Craig (1995, p.318 and p.341) about the Neyman's *factorization theorem.*

then not only $Q|\underline{S}$ and $R|\underline{S}$ but also $Q|X,\ Y$ and $R|X,\ Y$ have exactly the same Dirichlet distribution whose posterior density function can be written as

$$\pi(\theta, \rho \mid \underline{s}) = \begin{cases} \xi\theta^{\alpha+x-1}\rho^{\beta+y-1}(1-\theta-\rho)^{\gamma+n-x-y-1} & \text{if}(\theta, \rho) \in D, \\ 0 & \text{elsewhere.} \end{cases} \tag{19}$$

In the Bayesian statistics, all the information is involved in the posterior density function of *Equation* (19). In a similar manner to have the prior mean such as *Equation* (13), we are able to calculate posterior mean values for the guest who utilizes the sample data $\underline{s}$ of (quasi or actual) experiences to make it influenced to her decision:[6]

$$P_r(A \mid \underline{s}) = E\{E\{A \mid \underline{S}\}\} = E\{Q \mid \underline{S}\} = \frac{\alpha + x}{\alpha + \beta + \gamma + n}; \tag{20}$$

$$P_r(B \mid \underline{s}) = E\{E\{B \mid \underline{S}\}\} = E\{R \mid \underline{S}\} = \frac{\beta + y}{\alpha + \beta + \gamma + n}; \tag{21}$$

$$P_r(C \mid \underline{s}) = 1 - E\{Q \mid \underline{S}\} - E\{R \mid \underline{S}\} = \frac{\gamma + n - x - y}{\alpha + \beta + \gamma + n}. \tag{22}$$

Because we have $n=x=y=0$ in any *no data* problem, the posterior mean includes its corresponding prior mean. Let $\bar{\theta}$ be a maximum likelihood estimate for θ, then *Equation* (20) is a weighted average of the prior mean of *Equation* (13) and the estimate $\bar{\theta} = x/n$ as shown in Appendix, that is,

$$P_r(A \mid \underline{s}) = \frac{\alpha + \beta + \gamma}{\alpha + \beta + \gamma + n}\,\frac{\alpha}{\alpha + \beta + \gamma} + \frac{n}{\alpha + \beta + \gamma + n}\,\frac{x}{n}$$

[6] See, for example, Hogg and Craig (1995, p.367) about the posterior mean value, which minimizes a Bayes' risk of an expected square-error loss function given a sufficient statistics.

$$= \frac{\alpha+\beta+\gamma}{\alpha+\beta+\gamma+n} P_r(A) + \frac{n}{\alpha+\beta+\gamma+n} \bar{\theta}.$$

It can be seen that the posterior mean $P_r(A|\underline{s})$ converges to the estimate $\bar{\theta}$ as the sample size n goes up. So does $P_r(B|\underline{s})$ to the maximum likelihood estimate y/n, say $\bar{\rho}$ for the population parameter ρ. One may think that those estimates can be used for a proxy of the host's strategy about the red chip's hideaway as they can tell us at least if the events A, B, and C are equally likely or not.

Without losing any generality, say that after the guest has observed the sample $\underline{s}$, she picks up Box A as the first choice in Process 2] and the host shows Box B as a gray chip or $b=0$ in Process 3]. Let V and W be a random variable whose particular realization v and w respectively satisfies with

$$v \equiv \frac{\theta}{1-\rho} \text{ and } w \equiv \rho.$$

Since a *Jacobian*, J of the transformation: $\theta = v(1-\mathrm{w})$; and $\rho = w$ is given by

$$J \equiv \frac{\partial\theta}{\partial v}\frac{\partial\rho}{\partial w} - \frac{\partial\theta}{\partial w}\frac{\partial\rho}{\partial v} = (1-w)(1) - (-v)(0) = 1 - w > 0$$

and so *Equation* (19) can be transformed into a function h such that

$$h(v,\ w\,|\,\underline{s}) = J\pi(\theta,\ \rho\,|\,\underline{s}) = (1-w)\pi(v(1-w),\ w\,|\,\underline{s}) = \pi(v\,|\,\underline{s})\pi(w\,|\,\underline{s})$$

where

$$\pi(v\,|\,\underline{s}) = \begin{cases} \xi_v v^{\alpha+x-1}(1-v)^{\gamma+n-x-y-1} & \text{if } 0 < v < 1, \\ 0 & \text{elsewhere,} \end{cases} \tag{23}$$

$$\pi(w\,|\,\underline{s}) = \begin{cases} \xi_w v^{\beta+y-1}(1-w)^{\alpha+\gamma+n-y-1} & \text{if } 0 < w < 1, \\ 0 & \text{elsewhere,} \end{cases} \tag{24}$$

with constant coefficients

$$\xi_v \equiv \frac{\Gamma(\alpha+\gamma+n-y)}{\Gamma(\alpha+x)\Gamma(\gamma+n-x-y)} \text{ and } \xi_w \equiv \frac{\Gamma(\alpha+\beta+\gamma+n)}{\Gamma(\beta+y)\Gamma(\alpha+\gamma+n-y)},$$

not only $V|\underline{S}$ but $W|\underline{S}$ independently has a beta distribution whose probability density function is *Equation* (23) and *Equation* (24), respectively. Therefore, the guest, who has already observed data $\underline{s}$, expects that the conditional probability of the event A given a gray chip or $b=0$ should be

$$P_r(A \mid b=0,\ \underline{s}) = E\{E\{A \mid b=0\} \mid \underline{S}\} = E\left\{\frac{Q}{1-R} \mid \underline{S}\right\} = E\{V \mid \underline{S}\}$$

$$\equiv \int_0^1 v\pi(v \mid \underline{s})\,dv = \frac{\alpha+x}{\alpha+\gamma+n-y}, \tag{25}$$

which is a weighted average of a prior mean $\alpha/(\alpha+\gamma)$ of V or $Q/(1-R)$ and a maximum likelihood estimate $x/(n-y)$, say $\bar{v}$ of v as shown in Appendix, *i. e.*,

$$P_r(A \mid b=0,\ \underline{s}) = \frac{\alpha+\gamma}{\alpha+\gamma+n-y} E\{V\} + \frac{n+y}{\alpha+\gamma+n-y} \bar{v}.$$

One can see that as the sample size n goes up, *Equation* (25) converges to the estimate $\bar{v}$ no matter how tightly the guest might believe on the events of the red chip at the very beginning.[7] On the other hand, we have

$$P_r(C \mid b=0,\ \underline{s}) = 1 - P_r(A \mid b=0,\ \underline{s}) = \frac{\gamma+n-x-y}{\alpha+\gamma+n-y}. \tag{26}$$

Consequently, we are able to expand theorems discussed in the last

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[7] This tightness can be expressed by hyper parameters in *Equation*, (12): The larger they are, the more tightly the guest believes; Because $\alpha+\gamma=n-y$ holds on the middle point in this case, a convergence to $\bar{v}$ needs enough data that satisfies with $\alpha+\gamma<n-y$ at least.

section.

Theorem 1: *After the host shows that Box B has contained a gray chip in the* $(n+1)$*th game,* $P_r(A|\underline{s})$ *as well as* $P_r(C|\underline{s})$ *increase.*

Proof: Subtract *Equation* (20) from *Equation* (25), and we have

$$P_r(A \mid b=0,\ \underline{s}) - P_r(A \mid \underline{s}) = \frac{(\alpha+x)(\beta+y)}{(\alpha+\gamma+n-y)(\alpha+\beta+\gamma+n)} > 0\,.$$

In the meantime, subtracting *Equation* (22) from *Equation* (26) yields

$$P_r(C \mid b=0,\ \underline{s}) - P_r(C \mid \underline{s}) = \frac{(\gamma+n-x-y)(\beta+y)}{(\alpha+\gamma+n-y)(\alpha+\beta+\gamma+n)} > 0\,.$$

Recall that all the hyper parameters are positive. ***Q. E. D.***

Theorem 2: *After the host reveals Box B has had a gray chip in the* $(n+1)$*th game, the guest again falls into a dilemma if and only if* $P_r(A|\underline{s}) = P_r(C|\underline{s})$.

Proof: ⇒) Equalize *Equation* (25) to $\frac{1}{2}$ for the dilemma, and one has

$$\alpha + x = \gamma + n - x - y. \tag{27}$$

Divide both sides of *Equation* (27) by a positive $(\alpha+\beta+\gamma+\mathrm{n})$, and one has *Equation* (20) on the left as well as *Equation* (22) on the right.

⇐) Suppose that $P_r(A|\underline{s}) = P_r(C|\underline{s})$ at the outset of the game. As its common denominator of $(\alpha+\beta+\gamma+\mathrm{n})$ is positive, one has *Equation* (27). So, divide both sides of *Equation* (27) by a positive $(\alpha+\gamma+\mathrm{n}-y)$ this time, and one has *Equation* (25) on the left and *Equation* (26) on the right. ***Q. E. D.***

Theorem 3: *After the host reveals Box B has had a gray chip in the* $(n+1)$*th game, a large-small relation between* $P_r(A|\underline{s})$ *and* $P_r(C|\underline{s})$ *keeps order: That is, if* $P_r(A|\underline{s}) > P_r(C|\underline{s})$*, then* $P_r(A|b=0,\ \underline{s}) > P_r(C|b=0,\ \underline{s})$*; but if* $P_r(A|\underline{s}) < P_r(C|\underline{s})$*, then* $P_r(A|b=0,\ \underline{s}) < P_r(C|b=0,\ \underline{s})$.

Proof: Suppose that $P_r(A|\underline{s}) > P_r(C|\underline{s})$. Since $(\alpha + \beta + \gamma + n)$ is positive, one has $\alpha + x > \gamma + n - x - y$. Divide its both sides by $(\alpha + \gamma + n - y)$, and one has *Equation* (25) on the left and *Equation* (26) on the right. On the other hand, if $P_r(A|\underline{s}) < P_r(C|\underline{s})$, one has $\alpha + x < \gamma + n - x - y$. Divide it by $(\alpha + \gamma + n - y)$, and one has $P_r(A|b=0,\ \underline{s}) < P_r(C|b=0,\ \underline{s})$. ***Q. E. D.***

The Bayes' solution, derived from a modern analysis with a Bayes' risk of an expected square-error loss function, does not seem to go counter to one's intuition: Theorem $\underline{1}$ makes an appeal to the intuition that Box B is no longer a candidate for the red chip's whereabouts so that the other 2 probabilities should increase after the host's revelation of a gray chip; Theorem $\underline{2}$ appeals to the intuition that if the events that each box contains the red chip are truly equally likely, then the host's revelation of the gray chip should not make any different probabilities between the other 2 boxes left; Theorem $\underline{3}$ verifies the intuition that the host's revelation of the chip cannot be either the hint for the red one's whereabouts nor the advantage for the guest to revise her first hunch or prior knowledge, and she has not yet had another particular realization a so that on the second stage, she should act just according to what she has believed as her prior knowledge on the first stage. As discussed in Section 2, on the first stage she can employ one of 4 strategies based upon her belief to the knowledge. One may replace θ and ρ by *Equation* (20) and (21), respectively, or possibly by the maximum likelihood estimates $\bar{\theta}$ and $\bar{\rho}$, respectively because it has been shown that *Equation* (20) and (21) converges to it as the sample size n increases. Recall that $\bar{\theta} \equiv x/n$ and $\bar{\rho} \equiv y/n$. Let us write down the 4 strategies with those estimates.

Strategy $\underline{1}$: Truly equally likely; $\bar{\theta} = \bar{\rho} = 1 - \bar{\theta} - \bar{\rho} = \frac{1}{3}$.]

The guest chooses Box A because there is no difference among boxes. She falls into the dilemma on the second stage or in Process 4].

Strategy 2: Most likely; $\bar{\theta} \geq \mathrm{Max}\{\bar{\rho},\ 1- \bar{\theta} - \bar{\rho}\}$.]

The guest chooses Box A because this box has had the highest possibility to contain the red chip, so that she does not switch this box in Process 4].

Strategy 3: More likely; i) $1- \bar{\theta} - \bar{\rho} \leq \bar{\theta} < \bar{\rho}$ or ii) $\bar{\rho} \leq \bar{\theta} < 1 - \bar{\theta} - \bar{\rho}$.]

The guest chooses Box A as the second best. She switches it unless the host gets rid the most favorite box in her mind of the other 2 boxes.

Strategy 4: Least likely; $\bar{\theta} \leq \mathrm{Min}\{\bar{\rho},\ 1- \bar{\theta} - \bar{\rho}\}$.]

The guest chooses Box A even if she is sure that another box contains the red chip. So, she switches it no matter which box the host might reveal.

The strategies 2, 3, and 4 tell us that on the second stage, the guest eventually picks up the most likely box of the two remaining, which might contain the red chip. Suppose, for example, one observed a sample with size n=4,500 in which Box A contained the red chip 1,500 times but Box, B had it 1,200 times, then one has $\bar{\theta}$ $=1{,}500/4{,}500=\frac{1}{3}$ whereas $\bar{\rho}$ $=1{,}200/4{,}500=\frac{4}{15}$. In this example, she has taken Strategy 3 and so she will switch Box A to C, whose estimate is given by $\frac{2}{5}$ $(=1-\frac{1}{3}-\frac{4}{15})$. Needless to add, one is able to compute the exact solutions by *Equation* (20) through (22) with hyper parameters such as $\alpha = \beta = \gamma = 1$ to the Laplace's *insufficient reason*.

4. CONCLUDING REMARKS

In this paper, we have studied the host-guest game such as the Monty Hall dilemma and considered it as so-called the Bayesian statistics problem where any prior knowledge to a trial is treated as a random variable rather than merely as an unknown constant. Because there are 3 possible outcomes of winning a nice present, the game relates to a 3-point distribution with 2 population parameters of the guest's personal or subjective probabilities of winning the present. Since they depend upon the guest's prior knowledge or (quasi or actual) experiential values of random variables, we have assigned a Dirichlet distribution to them.

Owing to the tractability of a conjugate family as stated in Footnote 4, it has been shown that a Bayes' solution is the mean of a posterior (Dirichlet or beta) distribution given joint sufficient statistics of 2 population parameters. Because the Bayes' solution is a weighted average of a maximum likelihood estimate and the mean of a prior distribution, it goes to the estimate as the number of trials or a sample size n goes up. If the events A, B, and C that each box contains the winning present (red chip) are equally likely to a sample $\underline{s}$ with sufficiently large size n, then the mean of the posterior Dirichlet distribution or the probability of the event given $\underline{s}$ such as $P_r(A|\underline{s})$ will converge to $\frac{1}{3}$. And if so equally likely, the mean of the posterior beta distribution or the probability of the event given the h ost's revelation and $\underline{s}$ such as $P_r(A|b=0, \underline{s})$ will not go to $\frac{1}{3}$ but to $\frac{1}{2}$.

The Bayes' solution with the convergence to $\frac{1}{2}$ wipes off one's disgrace like Ph. D.'s (Still, Sachs, and Bobo) in Tierney (1991) and Erdös in Hoffman (1998) because the solution, which is the unique unbiased minimum variance estimator, does not run counter to one's intuition as seen in the paper: It makes an appeal to the intuition that the choice revealed by the host is no longer a candidate for the present's whereabouts so that the other probabilities should increase after it; It also appeals to the intuition that the host's revelation should make no difference between the other choices left if the events are equally likely; It verifies the intuition that the revelation should be neither a clue for the present's whereabouts nor an advantage for the guest to revise her first hunch or prior knowledge.

On the second stage, therefore, the guest had better act based upon what she has believed on the first stage. There are 4 strategies that she can employ for the host-guest game. Even if it is still open to question which strategy is the best for her, it does not matter so much to answer it when she is always allowed to switch because on the second stage with the allowance, she eventually picks up the most likely choice of the two remaining, which may contain the winning present. A selection averse to the optimal estimator may be an interesting topic for a future research.

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APPENDIX

This appendix provides a few maximum likelihood estimators of θ, ρ, and $v \equiv \theta/(1-\rho)$. Let (A_1, B_1), (A_2, B_2), $\cdots$, (A_n, B_n) *denote a sample with size n* from the following equation or *Equation* (1) of

$$f(a,\ b) = \begin{cases} \dfrac{1}{a!b!(1-a-b)!}\,\theta^a\rho^b(1-\theta-\rho)^{1-a-b} & \text{if } (a,\ b) \in \triangle, \\ 0 & \text{elsewhere,} \end{cases}$$

where $\triangle \equiv \{(a, b)|(0, 0), (1, 0), (0, 1)\}$. Since the probability that the sample has taken realizations of (a_1, b_1), (a_2, b_2), $\cdots$, (a_n, b_n) is computed by a product of $f(a_1, b_1)$, $f(a_2, b_2)$, $\cdots$, $f(a_n, b_n)$ or *Equation* (16) of

$$k_1 = \theta^x\rho^y(1-\theta-\rho)^{n-x-y}$$

where $x \equiv a_1 + a_2 + \cdots + a_n$ and $y \equiv b_1 + b_2 + \cdots + b_n$, we have a natural logarithmic likelihood function of

$$\ln k_1 = x\ln\theta + y\ln\rho + (n-x-y)\ln(1-\theta-\rho).$$

After taking partial derivatives with respect to θ as well as ρ, set them equal to

0, and we have a necessary condition of

$$\frac{\partial \ln k_1}{\partial \theta} = \frac{x}{\theta} - \frac{n-x-y}{1-\theta-\rho} = 0 \text{ and } \frac{\partial \ln k_1}{\partial \rho} = \frac{y}{\rho} - \frac{n-x-y}{1-\theta-\rho} = 0$$

or a system of simultaneous equations with respect to θ and ρ as

$$(n-x-y)\theta = x(1-\theta-\rho) \text{ and } (n-x-y)\rho = y(1-\theta-\rho).$$

Therefore, we have the maximum likelihood estimators for θ and ρ denoted by $\bar{\theta}$ and $\bar{\rho}$, respectively as follows:

$$\bar{\theta} = \frac{x}{n} \equiv \frac{\sum_{i=1}^{n} a_i}{n}; \text{ and } \bar{\rho} = \frac{y}{n} \equiv \frac{\sum_{i=1}^{n} b_i}{n}$$

since its sufficient condition holds due to second partial derivatives of ln k_1 with respect to θ and ρ as

$$\frac{\partial^2 \ln k_1}{\partial \theta^2} = -\frac{\alpha}{\theta^2} - \frac{n-\alpha-\beta}{(1-\theta-\rho)^2} < 0,$$

$$\frac{\partial^2 \ln k_1}{\partial \theta \partial \rho} = \frac{\partial^2 \ln k_1}{\partial \rho \partial \theta} = -\frac{n-\alpha-\beta}{(1-\theta-\rho)^2},$$

$$\frac{\partial^2 \ln k_1}{\partial \rho^2} = -\frac{\beta}{\rho^2} - \frac{n-\alpha-\beta}{(1-\theta-\rho)^2} < 0,$$

and due to a positive *Hessian* denoted by $|H_2|$ as

$$|H_2| \equiv \frac{\partial^2 \ln k_1}{\partial \theta^2}\frac{\partial^2 \ln k_1}{\partial \rho^2} - \frac{\partial^2 \ln k_1}{\partial \theta \partial \rho}\frac{\partial^2 \ln k_1}{\partial \rho \partial \theta} = \frac{n^2}{\bar{\theta}\bar{\rho}(1-\bar{\theta}-\bar{\rho})} > 0.$$

In a similar manner, let (A_1, B_1), (A_2, B_2), $\cdots$, (A_n, B_n) be the sample from *Equation* (6) or

$$f(a|b) = \begin{cases} \dfrac{(1-b)!}{a!(1-a-b)!} v^a (1-v)^{1-a-b} & \text{if } a = 0,\ 1-b; \\ 0 & \text{elsewhere,} \end{cases}$$

in which $v \equiv \theta/(1-\rho)$. Because the probability k_3 that the sample has had realizations of (a_1, b_1), (a_2, b_2), $\cdots$, (a_n, b_n) is calculated by a joint probability mass function or a product of $f(a_1|b_1)$, $f(a_2|b_2)$, $\cdots$, $f(a_n|b_n)$, that is,

$$k_3 \equiv v^x (1-v)^{n-x-y},$$

we have another natural logarithmic likelihood function ln k_3 as

$$\ln k_3 \equiv x \ln v + (n-x-y)\ln(1-v).$$

Since necessary and sufficient conditions for the maximization are obtained by

$$\frac{d \ln k_3}{dv} = \frac{x}{v} - \frac{n-x-y}{1-v} = 0 \text{ and so } (n-x-y)v = x(1-v),$$

$$\frac{d^2 \ln k_3}{dv^2} = -\frac{x}{v^2} - \frac{n-x-y}{(1-v)^2} < 0,$$

we have the maximum likelihood estimator $\bar{v}$ for v as

$$\bar{v} = \frac{x}{n-y} \equiv \frac{\sum_{i=1}^{n} a_i}{n-\sum_{i=1}^{n} b_i} = \frac{\sum_{i=1}^{n} a_i/n}{(n-\sum_{i=1}^{n} b_i)/n} = \frac{\bar{\theta}}{1-\bar{\rho}}.$$

Recall that $\theta \equiv P_r(A)$ and $\rho \equiv P_r(B)$, and consider the following examples: Suppose that one observed a sample with size $n=1$ that not only Box A but also Box B has not contained the red chip, then one has $\bar{\theta}=0/1=0$, $\bar{\rho}=0/1=0$, and $\bar{v}=0/(1-0)=0$; Suppose that one observed a sample with size $n=50$ that Box A contained the red chip 16 times but Box B had it 18 times, then one has $\bar{\theta}=16/50=8/25$, $\bar{\rho}=18/50=9/25$, and $\bar{v}=16/(50-18)=1/2$; Suppose that one observed a sample with size $n=4{,}500$ that Box A contained the red chip 1,500 times but Box B had it 1,200 times, then one has $\bar{\theta}=1{,}500/4{,}500=1/3$, $\bar{\rho}=1{,}200/4{,}500=4/15$, and $\bar{v}=1{,}500/(4{,}500-1{,}200)=5/11$; It can be seen that $\bar{\theta}=(n/3)/n=1/3$, $\bar{\rho}=(n/3)/n=1/3$, and

$$\bar{v} = \frac{x}{n-y} = \frac{n/3}{n-n/3} = \frac{n/3}{2n/3} = \frac{1}{2}$$

as long as those events are equally likely or that $x=y=\frac{n}{3}$ is actually met.