Savage on St. Petersburg Paradox

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1. Introduction

Leonard Jimmie Savage published *The Foundations of Statistics* in 1954. He referred to St. Petersburg Paradox retrospectively and discussed the solution by Daniel Bernoulli. This is a very interesting thing because Savage wrote the book from the viewpoint of Bayesian. As for St. Petersburg Paradox, William Feller solved it mathematically using sample theory of probability in 1937.

When you repeat the lottery n times, the total amount of prize money is like below supposing the prize of the kth trial is X_k.

$$S_n = X_1 + X_2 + \cdots + X_n$$

Feller proved

$$\Pr(\left|\frac{S_n}{G_n} - 1\right| > \varepsilon) \to 0$$

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if n is approaching ∞. Here G_n is the total amount of fair lottery fees. This means that when

$$\left|\frac{S_n}{G_n} - 1\right| < \varepsilon$$

then

$$\frac{S_n}{G_n} \to 1$$

Feller also showed the G_n that fitted to the condition is

$$G_n^* = an\log_2 n$$

Here a is the latch. Because of this, the fair lottery fee is finite as long as the number of trial is finite. Only when the number of trial is infinite, the fee also becomes infinite. A human being, however, cannot try it infinite times.

It is not clear if Savage knew the work of Feller. We can say that if he knew it, he would not accept the solution because Savage's standpoint is Bayesian theory of probability.

2. Savage and Expected Utility

Originally Savage worked as a mathematical assistant of von Neumann. He highly esteemed expected utility theory and make it the backbone of his book. In the book, he made a historical review of expected utility theory and wrote like this:[1]

Daniel Bernoulli（1700–1782), in the paper, seems to have been the first to point out that the principle is at best a rule of thumb, and he there suggested the maximization of expected utility as a more valid principle. Daniel Bernoulli's paper reproduces portions of a letter from Gabriel Cramer to Nicholas Bernoulli, which establishes Cramer's chronological priority to the idea of utility and most of the other main ideas of Bernoulli's paper. But it is Bernoulli's formulation together with some of the ideas that were specifically his that became popular and have had widespread influence to the present day. It is therefore appropriate to review Bernoulli's paper in some detail.

Then he advanced to refer to St. Petersburg paradox.[2]

Bernoulli cites a third, now very famous, example illustrating that men of prudence do not invariably obey the principle of mathematical expectation. This example, known as St. Petersburg paradox（because of the journal in which Bernoulli's paper was published）had earlier been publicized by Nicholas Bernoulli, and Daniel acknowledges it as the stimulus that led to his investigation of utility. Suppose, to state the St, Petersburg paradox succinctly, that a person could choose between an act leaving his wealth fixed at its present magnitude or one that would change his wealth at random, increasing it by $(2^n - f)$ dollars with probability 2^{-n} for every positive integer n. No matter how large the admission fee f may be, the expected income of the random act is infinite, as may easily be verified. Therefore, according to the principle of mathematical expectation, the random act is to be preferred to the status

1 Savage（1972), p.92.
2 Ibid., p.93.

quo. Numerical examples, however, soon convince any sincere person that he would prefer the status quo if *f* is at all large.

Everyone know that Bernoulli proposed the concave-shaped utility function here.[3]

Bernoulli went further than the law of diminishing marginal utility and suggested that the slope of utility as a function of wealth might, at least as a rule of thumb, be supposed, not only to decrease with, but to be inversely proportional to, the cash value of wealth. This, he pointed out, is equivalent to postulating that utility is equal to the logarithm (to any base) of the cash value of wealth. To this day, no other function has been suggested as a better prototype for Everyman's utility function. None the less, as Cramer pointed out in his aforementioned letter, the logarithm has a serious disadvantage; for, if the logarithm were the utility of wealth, the St. Petersburg paradox could be amended to produce a random act with an infinite expected utility (i.e., an infinite expected logarithm income) that, again, no one would really prefer to the status quo.

Savage thought that the weak point of Bernoulli's idea is a lack of probability.[4]

During the period when the probability-less idea of utility was popular with economists, they referred not only to the utility of money, but also to the utility of other consequences such as commodities (and services) and combinations (or, better, patterns of consumption) of commodities. The theory of choice

3 Ibid., p.94.
4 Ibid., pp.95–96.

among consequences was expressed by the idea that, among the available consequences, a person prefers those that have the highest utility for him. Also, the idea of diminishing marginal utility was extended from money to other commodities.

3. Savage's error

Bassett (1987) pointed that Savage committed an error here.[5]

> That the mathematician Cramer would have noted this in his letter seems very believable; the need for bounded utility is obvious to anyone who knows about logarithms and the mathematical expectation. To those who learned the early history from Savage there is nothing remarkable about this and no reason to think that events did not occur as Savage presented them. In fact, however, Cramer's letter does not say what Savage reports. There is no reference to the need for bounded utility in the letter reprinted in D. Bernoulli's paper. For some reason Savage made a mistake.

What is the reason at all? Basset continued like this:[6]

> To those who first learned the history from Savage the real surprise is not that Savage inadvertently gave too much credit to Cramer, but rather finding out when the need for bounded utility was apparently first noted. The plausible-sounding 1738 date for first observing the need for bounded utility turns out

5 Bassett (1987), p.518.

6 Ibid.

> to be wrong by two hundred years. The first person to make the bounded utility point was Karl Menger in 1934.

Bassett's answer is very simple. Because Savage knew that to solve the St. Petersburg paradox the bounded utility was necessary, he could not help finding the concept in Cramer's letter.[7]

> My current guess regarding Savage's mistake is that it arose because Savage knew that bounded utility was needed, he knew Cramer's letter mentioned a bounded utility function, and as he looked away from Bernoulli's paper the two thoughts intermingled and he ended up giving Cramer credit for more than what was in his letter. The mistake must have gone unrecognized because it is still in the 1972 edition of *Foundations.*

4. Conclusion

As already referring to in the introduction, Feller could avoid St. Petersburg paradox in 1937. He was a scholar for sample theory of probability. He could treat the paradox saying that only the infinite trials brought about an infinite admission fee. People, however, can try the lottery only finite times at most. Because of this reason, the paradox can be avoided.

Savage adopted Bayesian theory of probability. People can make trials infinite times subjectively. Logarithm-shaped utility function is not bounded. To solve the paradox, Savage, a Bayesian, need bounded utility function. As a result, Savage

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[7] Ibid., p.521–522.

did read mistakably bounded utility idea in Cramer's letter.

References

Bassett, G. W., Jr., 'The St. Petersburg paradox and bounded utility,' *History of Political Economy* 19: 4, Duke University Press, Durham, 1987.

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