The extremal rays of blow-ups of projective spaces at points

Hiroshi SATO

(Received November 30, 2019)

Abstract: In this paper, we study blow-ups of projective spaces at one, two, three or four torus invariant points. After some anti-flips, they become Fano varieties in some cases. We give the explicit description of extremal rays of them by using the notion of primitive relations.

2010 Mathematics Subject Classification: Primary 14M25; Secondary 14J45.

Key words and phrases: toric varieties, projective spaces, blow-ups, Fano varieties.

1. Introduction

In [VK], the following smooth toric Fano d-folds are studied: Let $\{e_1, \ldots, e_d\}$ be the standard basis for $\mathbb{Z}^d \subset \mathbb{R}^d$. The normal fan for the convex hull P of

$$\{\pm e_1, \ldots, \pm e_d, \pm(e_1 + \cdots + e_d)\}$$

determines the Gorenstein toric Fano d-fold X_P. X_P is smooth when d is even, and X_P looks like the blow-up of the d-dimensional projective space $\mathbb{P}^d$ at $d+1$ points. However, X_P is not a blow-up of $\mathbb{P}^d$ when $d \geq 4$, and have more complicated structures. In this case, there exists a sequence of anti-flips

$$X_n \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_r = X_P,$$

where X_n is the blow-up of $\mathbb{P}^d$ at $d+1$ torus invariant points, while $Y_1, \ldots, Y_{r-1}$ are smooth projective toric d-folds which are not weak Fano varieties. We remark that X_P is V^d in [VK].

Related to this phenomenon, in this paper, we investigate anti-flips of the blow-up X_n of $\mathbb{P}^d$ at n torus invariant points for $n \leq 4$. As results, we obtain even-dimensinal smooth toric Fano varieties. These smooth toric Fano varieties have a structure of a $(\mathbb{P}^1)^n$-bundle over $\mathbb{P}^{d-n}$ without the three exceptional cases (see X_3^{3+}, X_4^{6+} and Y_4^{4+} in Section 3) where these Fano varieties have similar structures to V^d.

Acknowledgments. The author was partially supported by JSPS KAKENHI Grant Number JP18K03262.

2. Blow-ups of projective spaces at points

In this section, we describe the blow-ups of $\mathbb{P}^d$ at n torus invariant points by using the toric geometry. We remark that there exist exactly $d+1$ torus invariant points in $\mathbb{P}^d$. Thus, we have $n \le d+1$. For the basic theory of the toric geometry, see [CLS], [F] and [O]. For the toric Mori theory, see [FS], [M] and [R] (see also [Ba1], [Ba2] and [S]).

Let $X = X_\Sigma$ be the smooth projective toric d-fold associated to a fan Σ in $N = \mathbb{Z}^d$. Put

$$\mathrm{G}(\Sigma) := \{\text{the primitive generaters for 1-dimensional cones in } \Sigma\} \subset N.$$

The following notion is very important for our theory.

Definition 2.1. A non-empty subset $P \subset \mathrm{G}(\Sigma)$ is a *primitive collction* in Σ (or X) if

(1) P does not generate a cone in Σ, while
(2) any proper subset of P generates a cone in Σ.

For a primitive collection $\{x_1, \ldots, x_l\}$, there exists a unique cone in Σ which contains $x_1 + \cdots + x_l$ in its relative interior. Let $\{y_1, \ldots, y_m\} \subset \mathrm{G}(\Sigma)$ be the generaters for this cone. Then, we have a linear relation

$$x_1 + \cdots + x_l = a_1 y_1 + \cdots + a_m y_m,$$

where $a_1, \ldots, a_m$ are positive integers. We call this relation the *primitive relations* for $\{x_1, \ldots, x_l\}$.

For any primitive collection P in X, we obtain a numerical 1-cycle on X by using its primitive relation. In particular, it is well-known that the numerical 1-cycles associated to primitive collections in X generate Mori cone $\mathrm{NE}(X)$. So, we say that a primitive collection is *extremal* if the associated 1-cycle generates an extremal ray of $\mathrm{NE}(X)$. We should remark that Σ can be recovered by all the primitive relations of Σ.

For blow-ups, the primitive collections can be calculated as follows:

Proposition 2.2. *Let $X = X_\Sigma$ be a smooth projective toric variety and $X' \to X$ be the blow-up with respect to a cone $\langle x_1, \ldots, x_r\rangle$ in Σ. Put $z := x_1 + \cdots + x_r$. Then, the primitive collections of X' are as follows:*

(1) $\{x_1, \ldots, x_r\}$.
(2) *Any primitive collection P in Σ such that $\{x_1, \ldots, x_r\} \not\subset P$.*
(3) *For any minimal element in*

$$\{P \setminus \{x_1, \ldots, x_r\} \mid P \text{ is a primitive collection in } \Sigma,\ P \cap \{x_1, \ldots, x_r\} \ne \emptyset\},$$

$$(P \setminus \{x_1, \ldots, x_r\}) \cup \{z\}.$$

By applying Proposition 2.2 n times, we can calculate the primitive relations of n points blow-up of $\mathbb{P}^d$.

Proposition 2.3. *Let $f : X_n \to \mathbb{P}^d$ be the toric blow-up of $\mathbb{P}^d$ at n torus invariant points for $1 \le n \le d+1$ and Σ_n the fan associated to X_n. Then, the primitive relations of Σ_n are as follows:*

$$x_i + y_i = 0\ (1 \le i \le n),\ x_1 + \cdots + \check{x}_i + \cdots + x_{d+1} = y_i\ (1 \le i \le n) \text{ and}$$

$$y_i + y_j = x_1 + \cdots + \check{x}_i + \cdots + \check{x}_j + \cdots + x_{d+1}\ (1 \le i < j \le n),$$

where $\mathrm{G}(\Sigma_n) = \{x_1, \ldots, x_{d+1}, y_1, \ldots, y_n\}$. In particular, Σ_n has exactly $\frac{n(n+3)}{2}$ primitive collections.

Conversely, we can calculate the primitive collections of the blow-down of a toric variety.

Proposition 2.4. *Let X be a smooth projective toric variety and $X \to \overline{X}$ the blow-down with respect to an extremal primitive relation*

$$x_1 + \cdots + x_m = z$$

of X. Then, the primitive collections of $\overline{X}$ are as follows:

(1) *Any primitive collection P in X such that $P \neq \{x_1, \ldots, x_m\}$ and $z \notin P$.*
(2) *For a primitive collection P in X such that $z \in P$ and $(P \setminus \{z\}) \cup S$ is not a primitive collection in X for any proper subset $S \subset \{x_1, \ldots, x_m\}$,*

$$(P \setminus \{z\}) \cup \{x_1, \ldots, x_m\}.$$

By using Propositions 2.2 and 2.4, we obtain the following. This theorem is essential in the calculations in Section 3.

Theorem 2.5. *Let $X = X_\Sigma$ and $X^+ = X_{\Sigma^+}$ be smooth projective toric varieties, and $X \dashrightarrow X^+$ the anti-flip with respect to an extremal primitive relation*

$$x_1 + \cdots + x_l = y_1 + \cdots + y_m,$$

where $\{x_1, \ldots, x_l, y_1, \ldots, y_m\} \subset \mathrm{G}(\Sigma)$. Then, the primitive collections of Σ^+ are as follows:

(1) *$\{y_1, \ldots, y_m\}$ whose primitive relation is*

$$y_1 + \cdots + y_m = x_1 + \cdots + x_l.$$

(2) *Any primitive collection P in Σ such that $\{y_1, \ldots, y_m\} \not\subset P$ and $P \neq \{x_1, \ldots, x_l\}$.*
(3) *For any minimal element in*

$$\{P \setminus \{y_1, \ldots, y_m\} \mid P \text{ is a primitive collection in } \Sigma,\ P \cap \{y_1, \ldots, y_m\} \neq \emptyset\}$$

such that $(P \setminus \{y_1, \ldots, y_m\}) \cup S$ does not contain a primitive collection for any proper subset $S \subset \{x_1, \ldots, x_l\}$,

$$(P \setminus \{y_1, \ldots, y_m\}) \cup \{x_1, \ldots, x_l\}.$$

Proof. X^+ is obtained by blowing-up X with respect to the cone $\langle y_1, \ldots, y_m \rangle$ and by blowing-down with respect to the extremal primitive relation

$$x_1 + \cdots + x_l = z,$$

where $z := y_1 + \cdots + y_m$. Therefore, we can apply Propositions 2.2 and 2.4. □

Remark 2.6. Obviously, Theorem 2.5 is available for flips and flops, too.

Here, we give the characterization of Fano varieties using the notion of primitive relations for the reader's convenience:

Proposition 2.7. *Let $X = X_\Sigma$ be a smooth projective toric variety. Then, X is Fano if and only if for any primitive relation*

$$x_1 + \cdots + x_l = a_1 y_1 + \cdots + a_m y_m$$

in Σ, $l - (a_1 + \cdots + a_m) > 0$ holds.

3. Anti-flips

In this section, we consider anti-flips of X_n in Proposition 2.3 in order to obtain Fano varieties. We deal with the cases where $n = 1, 2, 3$ and 4. For calculations of anti-flips, we use Theorem 2.5.

[I] 1 point blow-up. Proposition 2.3 says that the primitive relations of Σ_1 are

$$x_1 + y_1 = 0 \text{ and } x_2 + \cdots + x_{d+1} = y_1,$$

where $\mathrm{G}(\Sigma_1) = \{x_1, \ldots, x_{d+1}, y_1\}$. So, X_1 itself is a Fano variety (see [Bo]).

[II] 2 points blow-up. Proposition 2.3 says that the primitive relations of Σ_2 are

$$(2.1)\ x_1 + y_1 = 0,\ \ (2.2)\ x_2 + y_2 = 0,$$

$$(2.3)\ x_2 + x_3 + \cdots + x_{d+1} = y_1,\ \ (2.4)\ x_1 + x_3 + \cdots + x_{d+1} = y_2 \text{ and}$$

$$(2.5)\ y_1 + y_2 = x_3 + \cdots + x_{d+1},$$

where $\mathrm{G}(\Sigma_2) = \{x_1, \ldots, x_{d+1}, y_1, y_2\}$. One can easily see that X_2 is Fano if and only if $d = 2$. If $d = 3$, then X_2 has a flopping contraction. So, let $d \geq 4$.

Only we have to do is to do the anti-flip with respect to the primitive relation (2.5). Let $X_2 \dashrightarrow X_2^+$ be this anti-flip. Then, the primitive relations of X_2^+ are

$$(2.1)\ x_1 + y_1 = 0,\ \ (2.2)\ x_2 + y_2 = 0 \text{ and } (2.5^+)\ x_3 + \cdots + x_{d+1} = y_1 + y_2.$$

X_2^+ is a toric Fano variety. Thus, we obtain the following:

Theorem 3.1. *Let X_2 be a blow-up of $\mathbb{P}^d$ at 2 torus invariant points. Then, the following hold:*

(1) *X_2 is Fano if and only if $d = 2$.*
(2) *If $d \geq 4$, then after one anti-flip, we obtain a Fano variety X_2^+. Moreover, X_2^+ is a $(\mathbb{P}^1 \times \mathbb{P}^1)$-bundle over $\mathbb{P}^{d-2}$.*

[III] 3 points blow-up. Proposition 2.3 says that the primitive relations of Σ_3 are

$$(3.1)\ x_1 + y_1 = 0,\ \ (3.2)\ x_2 + y_2 = 0,\ \ (3.3)\ x_3 + y_3 = 0,$$

$$(3.4)\ x_2 + x_3 + x_4 + \cdots + x_{d+1} = y_1,\ \ (3.5)\ x_1 + x_3 + x_4 + \cdots + x_{d+1} = y_2,$$

$$(3.6)\ x_1 + x_2 + x_4 + \cdots + x_{d+1} = y_3,$$

$$(3.7)\ y_1 + y_2 = x_3 + x_4 + \cdots + x_{d+1},\ \ (3.8)\ y_1 + y_3 = x_2 + x_4 + \cdots + x_{d+1} \text{ and}$$

$$(3.9)\ y_2 + y_3 = x_1 + x_4 + \cdots + x_{d+1},$$

where $\mathrm{G}(\Sigma_3) = \{x_1, \ldots, x_{d+1}, y_1, y_2, y_3\}$. One can easily see that X_3 is Fano if and only if $d = 2$. If $d = 3$, then X_3 has flopping contractions. So, let $d \geq 4$.

First of all, we do 3 anti-flips φ_1, φ_2 and φ_3 with respect to (3.7), (3.8) and (3.9), respectively. Let

$$X_3 \overset{\varphi_1}{\dashrightarrow} X_3^{1+} \overset{\varphi_2}{\dashrightarrow} X_3^{2+} \overset{\varphi_3}{\dashrightarrow} X_3^{3+}$$

be the sequence of the anti-flips. Then, we have the following:

- The primitive relations of X_3^{1+} are

$$(3.7^+)\ x_3 + x_4 + \cdots + x_{d+1} = y_1 + y_2,$$

$$(3.1),\ (3.2),\ (3.3),\ (3.6),\ (3.8) \text{ and } (3.9).$$

- The primitive relations of X_3^{2+} are

$$(3.8^+)\ x_2 + x_4 + \cdots + x_{d+1} = y_1 + y_3,$$
$$(3.1),\ (3.2),\ (3.3),\ (3.7^+) \text{ and } (3.9).$$

- The primitive relations of X_3^{3+} are

$$(3.9^+)\ x_1 + x_4 + \cdots + x_{d+1} = y_2 + y_3,$$
$$(3.1)\ x_1 + y_1 = 0,\ (3.2)\ x_2 + y_2 = 0,\ (3.3)\ x_3 + y_3 = 0,$$
$$(3.7^+)\ x_3 + x_4 + \cdots + x_{d+1} = y_1 + y_2,\ (3.8^+)\ x_2 + x_4 + \cdots + x_{d+1} = y_1 + y_3 \text{ and}$$
$$(3.10)\ y_1 + y_2 + y_3 = x_4 + \cdots + x_{d+1}.$$

X_3^{3+} is Fano if $d = 4$. This toric Fano 4-fold is of type M_1 in the list of [Ba2] (see also [S]). If $d = 5$, then X_3^{3+} has a flopping contraction. So, let $d \geq 6$.

Next, we do the anti-flip $\varphi_4 : X_3^{3+} \dashrightarrow Y_3^+$ with respect to (3.10). The primitive relations of Y_3^+ are

$$(3.10^+)\ x_4 + \cdots + x_{d+1} = y_1 + y_2 + y_3,$$
$$(3.1)\ x_1 + y_1 = 0,\ (3.2)\ x_2 + y_2 = 0 \text{ and } (3.3)\ x_3 + y_3 = 0.$$

Y_3^+ is a Fano variety. Thus, we have the following:

Theorem 3.2. *Let X_3 be a blow-up of $\mathbb{P}^d$ at 3 torus invariant points. Then, the following hold:*

(1) *X_3 is Fano if and only if $d = 2$.*
(2) *If $d = 4$, then after 3 anti-flips, we obtain a Fano variety X_3^{3+}.*
(3) *If $d \geq 6$, then after 4 anti-flips, we obtain a Fano variety Y_3^+. Moreover, Y_3^+ is a $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$-bundle over $\mathbb{P}^{d-3}$.*

[IV] 4 points blow-up. Proposition 2.3 says that the primitive relations of Σ_4 are

$$(4.1)\ x_1 + y_1 = 0,\ (4.2)\ x_2 + y_2 = 0,\ (4.3)\ x_3 + y_3 = 0,\ (4.4)\ x_4 + y_4 = 0,$$
$$(4.5)\ x_2 + x_3 + x_4 + x_5 + \cdots + x_{d+1} = y_1,\ (4.6)\ x_1 + x_3 + x_4 + x_5 + \cdots + x_{d+1} = y_2,$$
$$(4.7)\ x_1 + x_2 + x_4 + x_5 + \cdots + x_{d+1} = y_3,\ (4.8)\ x_1 + x_2 + x_3 + x_5 + \cdots + x_{d+1} = y_4,$$
$$(4.9)\ y_1 + y_2 = x_3 + x_4 + x_5 + \cdots + x_{d+1},\ (4.10)\ y_1 + y_3 = x_2 + x_4 + x_5 + \cdots + x_{d+1},$$
$$(4.11)\ y_1 + y_4 = x_2 + x_3 + x_5 + \cdots + x_{d+1},\ (4.12)\ y_2 + y_3 = x_1 + x_4 + x_5 + \cdots + x_{d+1},$$
$$(4.13)\ y_2 + y_4 = x_1 + x_3 + x_5 + \cdots + x_{d+1} \text{ and } (4.14)\ y_3 + y_4 = x_1 + x_2 + x_5 + \cdots + x_{d+1},$$

where $\mathrm{G}(\Sigma_4) = \{x_1, \ldots, x_{d+1}, y_1, y_2, y_3, y_4\}$. If $d = 3$, then X_4 has flopping contractions. So, let $d \geq 4$.

First of all, we do 6 anti-flips φ_1, φ_2, φ_3, φ_4, φ_5 and φ_6 with respect to (4.9), (4.10), (4.11), (4.12), (4.13) and (4.14), respectively. Let

$$X_4 \overset{\varphi_1}{\dashrightarrow} X_4^{1+} \overset{\varphi_2}{\dashrightarrow} X_4^{2+} \overset{\varphi_3}{\dashrightarrow} X_4^{3+} \overset{\varphi_4}{\dashrightarrow} X_4^{4+} \overset{\varphi_5}{\dashrightarrow} X_4^{5+} \overset{\varphi_6}{\dashrightarrow} X_4^{6+}$$

be the sequence of the anti-flips. Then, we have the following:

- The primitive relations of X_4^{1+} are

$$(4.9^+)\ x_3 + x_4 + x_5 + \cdots + x_{d+1} = y_1 + y_2,$$
$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.7),\ (4.8),\ (4.10),\ (4.11),\ (4.12),\ (4.13) \text{ and } (4.14).$$

- The primitive relations of X_4^{2+} are

$$(4.10^+)\ x_2 + x_4 + x_5 + \cdots + x_{d+1} = y_1 + y_3,$$
$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.8),\ (4.9^+),\ (4.11),\ (4.12),\ (4.13) \text{ and } (4.14).$$

- The primitive relations of X_4^{3+} are

$$(4.11^+)\ x_2+x_3+x_5+\cdots+x_{d+1}=y_1+y_4,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.9^+),\ (4.10^+),\ (4.12),\ (4.13) \text{ and } (4.14).$$

- The primitive relations of X_4^{4+} are

$$(4.12^+)\ x_1+x_4+x_5+\cdots+x_{d+1}=y_2+y_3,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.9^+),\ (4.10^+),\ (4.11^+),\ (4.13),\ (4.14) \text{ and}$$

$$(4.15)\ y_1+y_2+y_3=x_4+x_5+\cdots+x_{d+1}.$$

- The primitive relations of X_4^{5+} are

$$(4.13^+)\ x_1+x_3+x_5+\cdots+x_{d+1}=y_2+y_4,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.9^+),\ (4.10^+),\ (4.11^+),\ (4.12^+),\ (4.14),\ (4.15) \text{ and}$$

$$(4.16)\ y_1+y_2+y_4=x_3+x_5+\cdots+x_{d+1}.$$

- The primitive relations of X_4^{6+} are

$$(4.14^+)\ x_1+x_2+x_5+\cdots+x_{d+1}=y_3+y_4,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.9^+),\ (4.10^+),\ (4.11^+),\ (4.12^+),\ (4.13^+),\ (4.15),\ (4.16),$$

$$(4.17)\ y_1+y_3+y_4=x_2+x_5+\cdots+x_{d+1} \text{ and } (4.18)\ y_2+y_3+y_4=x_1+x_5+\cdots+x_{d+1}.$$

X_4^{6+} is Fano if $d=4$. In this case, X_4^{6+} is isomorphic to $\widetilde{V}^4$ in [E] (see also [Ba2], [S] and [VK]). If $d=5$, then X_4^{6+} has flopping contractions. So, let $d\geq 6$.

Next, we do 4 anti-flips φ_7, φ_8, φ_9 and φ_{10} with respect to (4.15), (4.16), (4.17) and (4.18), respectively. Let

$$X_4^{6+} \overset{\varphi_7}{\dashrightarrow} Y_4^{1+} \overset{\varphi_8}{\dashrightarrow} Y_4^{2+} \overset{\varphi_9}{\dashrightarrow} Y_4^{3+} \overset{\varphi_{10}}{\dashrightarrow} Y_4^{4+}$$

be the sequence of the anti-flips. Then, we have the following:

- The primitive relations of Y_4^{1+} are

$$(4.15^+)\ x_4+x_5+\cdots+x_{d+1}=y_1+y_2+y_3,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.11^+),\ (4.13^+),\ (4.14^+),\ (4.16),\ (4.17) \text{ and } (4.18).$$

- The primitive relations of Y_4^{2+} are

$$(4.16^+)\ x_3+x_5+\cdots+x_{d+1}=y_1+y_2+y_4,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.14^+),\ (4.15^+),\ (4.17) \text{ and } (4.18).$$

- The primitive relations of Y_4^{3+} are

$$(4.17^+)\ x_2+x_5+\cdots+x_{d+1}=y_1+y_3+y_4,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.15^+),\ (4.16^+) \text{ and } (4.18).$$

- The primitive relations of Y_4^{4+} are

$$(4.18^+)\ x_1+x_5+\cdots+x_{d+1}=y_2+y_3+y_4,$$

$$(4.1),\ (4.2),\ (4.3),\ (4.4),\ (4.15^+),\ (4.16^+),\ (4.17^+) \text{ and}$$

$$(4.19)\ y_1+y_2+y_3+y_4=x_5+\cdots+x_{d+1}.$$

Y_4^{4+} is Fano if $d = 6$. If $d = 7$, then Y_4^{4+} has a flopping contraction. So, let $d \geq 8$.

Finally, we do the anti-flip $\varphi_{11} : Y_4^{4+} \dashrightarrow Z_4^+$ with respect to the primitive relation (4.19). The primitive relations of Z_4^+ are

$$(4.19^+)\ x_5 + \cdots + x_{d+1} = y_1 + y_2 + y_3 + y_4,$$

$$(4.1),\ (4.2),\ (4.3) \text{ and } (4.4).$$

Thus, we can easily see that Z_4^+ is a Fano variety, and obtain the following:

Theorem 3.3. *Let X_4 be a blow-up of $\mathbb{P}^d$ at 4 torus invariant points. Then, the following hold:*

(1) *If $d = 4$, then after 6 anti-flips, we obtain a Fano variety X_4^{6+}.*
(2) *If $d = 6$, then after 10 anti-flips, we obtain a Fano variety Y_4^{4+}.*
(3) *If $d \geq 8$, then after 11 anti-flips, we obtain a Fano variety Z_4^+. Moreover, Z_4^+ is a $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$-bundle over $\mathbb{P}^{d-4}$.*

By considering Theorems 3.1, 3.2 and 3.3, we end this section by giving the following conjecture:

Conjecture 3.4. For the blow-up X_n of $\mathbb{P}^d$ at n torus invariant points, the following hold: Put $d = 2e$ when d is even, while $d = 2e + 1$ when d is odd for a positive integer e.

(1) If $n \leq e$, then after anti-flips, we obtain a smooth toric Fano variety which has a structure of $(\mathbb{P}^1)^n$-bundle over $\mathbb{P}^{d-n}$.
(2) If $e + 1 \leq n \leq d + 1$ and d is even, then after anti-flips, we obtain a smooth toric Fano variety which does not admit any bundle structure.

References

[Ba1] V. Batyrev, On the classification of smooth projective toric varieties, Tohoku Math. J. **43** (1991), no. 4, 569–585.

[Ba2] V. Batyrev, On the classification of toric Fano 4-folds, Algebraic geometry, 9. J. Math. Sci. (New York) **94** (1999), 1021–1050.

[Bo] L. Bonavero, Toric varieties whose blow-up at a point is Fano, Tohoku Math. J. **54** (2002), no. 4, 593–597.

[CLS] D. A. Cox, J. B. Little, H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, **124**. American Mathematical Society, Providence, RI, 2011.

[E] G. Ewald, On the classification of toric Fano varieties, Discrete Comput. Geom. **3** (1988), 49–54.

[FS] O. Fujino and H. Sato, Introduction to the toric Mori theory, Michigan Math. J. **52** (2004), no. 3, 649–665.

[F] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, **131**. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[M] K. Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.

[O] T. Oda, *Convex bodies and algebraic geometry, An introduction to the theory of toric varieties*, Translated from the Japanese, Results in Mathematics and Related Areas (3) **15**, Springer-Verlag, Berlin, 1988.

[R] M. Reid, Decomposition of toric morphisms, *Arithmetic and geometry, Vol. II*, 395–418, Progr. Math. **36**, Birkhäuser Boston, Boston, MA, 1983.

[S] H. Sato, Toward the classification of higher-dimensional toric Fano varieties, Tohoku Math. J. **52** (2000), no. 3, 383–413.

[VK] V. E. Voskresenskij and A. A. Klyachko, Toroidal Fano varieties and root systems, Math. USSR-Izv. **24** (1985), 221–244.

Department of Applied Mathematics, Faculty of Sciences, Fukuoka University, 8-19-1, Nanakuma, Jonan-ku, Fukuoka 814-0180, Japan

Email address: hirosato@fukuoka-u.ac.jp