Some remarks on invariant rings under the actions of reflection groups related to Weyl groups

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Abstract We will consider some invariant rings over the finite field $\mathbb{F}_2$ and the field of rational numbers $\mathbb{Q}$. In particular, a few remarks on invariant rings under the reflection groups related to $W(Sp(5))$ for $\mathbb{F}_2$ and to $W(PU(3))$ for $\mathbb{Q}$ will be given.

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In [5], we have considered the mod 2 invariant rings related to the symplectic groups, namely $H^*(BT^n;\mathbb{F}_2)^{\overline{W_n}}$ where $\overline{W_n}$ is the mod 2 reduction of the integral representation $W_n = \phi_2 W(Sp(n))\phi_2^{-1}$. Here note that the reflection group $W(Sp(n))$ is generated by the permutation matrices Σ_n and the $n \times n$ diagonal matrix $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$, and that

$$\phi_2 = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \frac{-1}{2} & \cdots & \frac{-1}{2} & \frac{1}{2} \end{pmatrix}.$$

It is shown [5] that the invariant ring $H^*(BT^n;\mathbb{F}_2)^{\overline{W_n}}$ is polynomial for $n = 3, 4$, and that it is not polynomial for $n = 6, 8$. Thus, for instance, the case of $n = 5$ is remained open due to a heavy calculation involved. This time we have got a help from the Maxima software 5.41.0, [8]. Let W^* denote the dual representation of a subgroup W of $GL(n,\mathbb{F}_p)$. We will see that both of the invariant rings $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}}$ and $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}^*}$ are polynomial.

Theorem 1 *The invariant rings* $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}}$ *and* $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}^*}$ *are polynomial of the following types:*

(a) $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}} = \mathbb{F}_2[x_2, x_8, x_{12}, x_{16}, x_{20}]$.

(b) $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}^*} = \mathbb{F}_2[y_4, y_6, y_8, y_{10}, y_{32}]$.

The explicit expressions for the generators will be given in §1. Meantime, we note that $|\overline{W_n}| = \frac{|W(Sp(n))|}{2}$ so that $|\overline{W_5}| = 2^4 \cdot 5!$, since it is important to find a system of parameters, [11, Proposition 5.5.5].

Next we consider the cohomology in rational coeffients. Recall that for any ring R we have $H^*(BT^n;R) = R[t_1, t_2, \cdots, t_n]$ and that for a free R–module V of rank n we have $S(V) = H^*(BT^n;R)$. Consider the case of $n = 2$ and $R = \mathbb{Z}$. In §2 we will show an example such that $S(V)^W \otimes \mathbb{Q} = \mathbb{Q}[\alpha,\beta]$ and $\mathbb{Q}[\alpha,\beta] \cap S(V)^W \neq \mathbb{Z}[\alpha,\beta]$.

Some results related to this work were announced at a regional meeting of the Japan Math. Soc., [6]. And Duan announced some related work at The 2nd Pan-Pacific International Conference on Topology and Applications, [1].

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1 Modular invariant rings for $n = 5$

We will determine the structures of both invariant rings $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}}$ and $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}^*}$. The theory of invariant rings can be found in [7], [9], [11], [12] and [13]. First we give the explicit expressions for the generators $\{x_i\}$ for $H^*(BT^5;\mathbb{F}_2)^{\overline{W_5}}$.

We notice that $\overline{W_5}$ is generated by the symmetric group Σ_4 together with two reflections:

$$\overline{W_5} = \left\langle \Sigma_4, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle .$$

Let c_i $(1 \le i \le 4)$ be the fundamenthal symmetric polynomials on 4 letters, or the Chern classes for $H^*(BU(4);\mathbb{F}_2)$.

$$\begin{aligned} c_1 &= t_1 + t_2 + t_3 + t_4 \\ c_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 \\ c_3 &= t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 \\ c_4 &= t_1t_2t_3t_4 \end{aligned}$$

The follwing set V is invariant under the action of $\overline{W_5}$.

$$\begin{aligned} V = \{&t_1+t_2+t_3+t_4,\ t_1+t_2+t_3+t_4+t_5,\ t_1+t_2+t_3, t_1+t_2+t_4, \\ &t_1+t_3+t_4,\ t_2+t_3+t_4,\ t_1+t_2+t_3+t_5,\ t_1+t_2+t_4+t_5, \\ &t_1+t_3+t_4+t_5,\ t_2+t_3+t_4+t_5\} \end{aligned}$$

Consequently, each coefficient of the follwing polynomial $f(X)$ is $\overline{W_5}$–invariant. The underlined elements are the generators mensioned in Theorem 1.

$$\begin{aligned} f(X) = \prod_{v\in V}(X+v) =& X^{10} + \underline{x_2}X^9 + \underline{x_8}X^6 + x_2x_8X^5 + \underline{x_{12}}X^4 + x_2^3(x_2^4+x_8)X^3 \\ &+ \underline{x_{16}}X^2 + x_2(x_2^8 + x_2^4x_8 + x_2^2x_{12} + x_{16})X + \underline{x_{20}} \end{aligned}$$

For example, x_8 can be written in terms of t_i $(1 \le i \le 5)$ as follows:

$$\begin{aligned} x_8 = &t_5^4 + t_3t_4t_5^2 + t_2t_4t_5^2 + t_1t_4t_5^2 + t_2t_3t_5^2 + t_1t_3t_5^2 + t_1t_2t_5^2 + t_3t_4^2t_5 \\ &+ t_2t_4^2t_5 + t_1t_4^2t_5 + t_3^2t_4t_5 + t_2^2t_4t_5 + t_1^2t_4t_5 + t_2t_3^2t_5 + t_1t_3^2t_5 + t_2^2t_3t_5 \\ &+ t_1^2t_3t_5 + t_1t_2^2t_5 + t_1^2t_2t_5 + t_3^2t_4^2 + t_2^2t_4^2 + t_1^2t_4^2 + t_2^2t_3^2 + t_1^2t_3^2 + t_1^2t_2^2 \end{aligned}$$

If we use c_i $(1 \le i \le 4)$, we get the follwing.

$$
\begin{aligned}
x_2 &= t_5\\
x_8 &= t_5^4 + c_2t_5^2 + (c_1c_2 + c_3)t_5 + c_2^2\\
x_{10} &= t_5^5 + c_2t_5^3 + (c_1c_2 + c_3)t_5^2 + c_2^2t_5 = x_2x_8\\
x_{12} &= c_2t_5^4 + c_2^2t_5^2 + (c_1^3c_2 + c_1^2c_3 + c_1c_2^2 + c_1c_4 + c_2c_3)t_5 + (c_1c_2 + c_3)^2\\
x_{14} &= c_2t_5^5 + (c_1c_2 + c_3)t_5^4 + c_2^2t_5^3 = x_2^3(x_2^4 + x_8)\\
x_{16} &= (c_1c_2 + c_3)t_5^5 + (c_1^4 + c_1^2c_2 + c_4)t_5^4 + (c_1c_2^2 + c_2c_3)t_5^3\\
&\quad + (c_1^4c_2 + c_2c_4 + c_3^2)t_5^2 + (c_1^5c_2 + c_1^4c_3 + c_3c_4)t_5 + (c_1^4 + c_1^2c_2 + c_4)^2\\
x_{18} &= (c_1^2c_2 + c_1^4 + c_4)t_5^5 + (c_1^2c_3 + c_1^3c_2 + c_1c_4)t_5^4 + (c_2c_4 + c_1^4c_2 + c_1^2c_2^2)t_5^3\\
&\quad + (c_1^5c_2 + c_1^4c_3 + c_3c_4)t_5^2 + (c_1^8 + c_1^4c_2^2 + c_4^2)t_5\\
&= x_2(x_2^8 + x_2^4x_8 + x_2^2x_{12} + x_{16})\\
x_{20} &= c_1(c_1^2c_2 + c_1c_3 + c_4)t_5^5 + c_1c_2(c_1^2c_2 + c_1c_3 + c_4)t_5^3\\
&\quad + c_1(c_1^3c_2^2 + c_1c_3^2 + c_1c_2c_4 + c_3c_4)t_5^2\\
&\quad + c_1(c_1^6c_2 + c_1^5c_3 + c_1^4c_2^2 + c_1^4c_4 + c_1^3c_2c_3 + c_1c_3c_4 + c_4^2)t_5\\
&\quad + c_1^2(c_1^2c_2 + c_1c_3 + c_4)^2
\end{aligned}
$$

Proof of Theorem 1 (a) It is enough to show that the set $\{x_2, x_8, x_{12}, x_{16}, x_{20}\}$ is a system of parameters. Suppose $x_2 = 0,\ x_8 = 0,\ x_{12} = 0,\ x_{16} = 0\ ,\ x_{20} = 0$. It follows that $t_5 = 0,\ c_2^2 = 0,\ (c_1c_2 + c_3)^2 = 0,\ (c_1^4 + c_1^2c_2 + c_4)^2 = 0,\ c_1^2(c_1^2c_2 + c_1c_3 + c_4)^2 = 0$, and then $t_5 = 0,\ c_2 = 0,\ c_1c_2 + c_3 = 0,\ c_1^4 + c_1^2c_2 + c_4 = 0,\ c_1(c_1^2c_2 + c_1c_3 + c_4) = 0$. Furthermore, we see $t_5 = 0,\ c_2 = 0,\ c_3 = 0,\ c_1^4 + c_4 = 0,\ c_1c_4 = 0$. If $c_1 = 0$, then $c_4 = 0$ from $c_1^4 + c_4 = 0$. And $c_4 = 0$ implies that $c_1 = 0$ from $c_1^4 + c_4 = 0$. We obtain $c_1 = c_2 = c_3 = c_4 = 0$, and hence $t_i = 0$ for $i = 1, \ldots, 5$. This completes the proof the part (a) of Theorem 1. □

Next we give the explicit expressions for the generators $\{y_i\}$ for the dual case $H^*(BT^5; \mathbb{F}_2)^{\overline{W_5}^*}$. The argument is similar to the part (a) of Theorem 1.

$$
\begin{aligned}
c_1 &= t_1 + t_2 + t_3 + t_4\\
c_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4\\
c_3 &= t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4\\
c_4 &= t_1t_2t_3t_4
\end{aligned}
$$

$$
\begin{aligned}
y_4 &:= c_1^2 + c_2\\
y_6 &:= c_1c_2 + c_3\\
y_8 &:= c_1c_3 + c_4\\
y_{10} &:= c_1c_4
\end{aligned}
$$

The follwing set V is invariant under the action of $\overline{W_5}^*$.

$V = \{t_1{+}t_2, t_1{+}t_3,\ t_1{+}t_4, t_2{+}t_3,\ t_2{+}t_4,\ t_3{+}t_4,\ t_1{+}t_2{+}t_3,\ t_1{+}t_2{+}t_4,\ t_1{+}t_3{+}t_4,\ t_2{+}t_3{+}t_4\}$

Consequently, each coefficient of the follwing polynomial $f(X)$ is $\overline{W_5}^*$–invariant. The underlined elements are the generators mensioned in Theorem 1.

$$
\begin{aligned}
f(X) = \prod_{v\in V}(X + v) = X^{10} + \underline{y_4}X^8 + \underline{y_6}X^7 + \left(\underline{y_8} + y_4^2\right)X^6 + \underline{y_{10}}X^5 + \left(y_4^3 + y_6^2\right)X^4\\
+ \left(y_4^2y_6\right)X^3 + \left(y_4^2y_8 + y_4y_6^2 + y_6y_{10}\right)X^2 + \left(y_4^2y_{10} + y_6^3\right)X\\
+ y_4^2 + y_6^2 + y_4y_6y_{10} + y_6^2y_8 + y_{10}^2
\end{aligned}
$$

The follwing set U is also invariant under the action of $\overline{W_5}^*$.

$$\begin{aligned}U =\{&t_5,\ t_1+t_5,\ t_2+t_5,\ t_3+t_5,\ t_4+t_5,\ t_1+t_2+t_5,\ t_1+t_3+t_5,\ t_1+t_4+t_5,\ t_2+t_3+t_5,\\ &t_2+t_4+t_5,\ t_3+t_4+t_5,\ t_1+t_2+t_3+t_5,\ t_1+t_2+t_4+t_5,\ t_1+t_3+t_4+t_5,\\ &t_2+t_3+t_4+t_5,\ t_1+t_2+t_3+t_4+t_5\}\end{aligned}$$

$$\begin{aligned}g(X) = \prod_{u\in U}(X+u) = X^{16} &+ \left(y_4^4+y_6y_{10}+y_8^2\right)X^8 + \left(y_4^2y_6y_{10}+y_4^2y_8^2+y_4y_{10}^2+y_6^4+y_6y_8y_{10}\right)X^4\\ &+ \left(y_4^2y_{10}^2+y_4y_6y_8y_{10}+y_6^3y_{10}+y_6^2y_8^2+y_8y_{10}^2\right)X^2\\ &+ \left(y_6^2y_8y_{10}+y_4y_6y_{10}^2+y_{10}^3\right)X + \underline{y_{32}}\end{aligned}$$

$$\begin{aligned}y_{32} &= t_5^{16} + (y_6y_{10}+y_8^2+y_4^4)t_5^8 + (y_4y_{10}^2+y_6y_8y_{10}+y_4^2y_6y_{10}+y_4^2y_8^2+y_6^4)t_5^4\\ &\quad + (y_8y_{10}^2+y_4^2y_{10}^2+y_6y_8y_4y_{10}+y_6^3y_{10}+y_6^2y_8^2)t_5^2 + (y_{10}^3+y_6y_4y_{10}^2+y_6^2y_8y_{10})t_5\\ c_{5,4} &= y_{32}+y_4^8+y_6^2y_{10}^2+y_8^4\end{aligned}$$

Here $c_{5,4}$ is the Dickson invariant. In general, we see the following: $\mathbb{F}_2[t_1,t_2,\cdots,t_n]^{GL(n,\mathbb{F}_2)} = \mathbb{F}_2[c_{n,n-1},c_{n,n-2},\cdots,c_{n,0}]$ with $d(c_{n,i}) = 2^n - 2^i$, [7, §16-5].

$$c_{5,4} = \frac{\begin{vmatrix} t_1 & t_2 & t_3 & t_4 & t_5\\ t_1^2 & t_2^2 & t_3^2 & t_4^2 & t_5^2\\ t_1^4 & t_2^4 & t_3^4 & t_4^4 & t_5^4\\ t_1^8 & t_2^8 & t_3^8 & t_4^8 & t_5^8\\ t_1^{32} & t_2^{32} & t_3^{32} & t_4^{32} & t_5^{32}\end{vmatrix}}{\begin{vmatrix} t_1 & t_2 & t_3 & t_4 & t_5\\ t_1^2 & t_2^2 & t_3^2 & t_4^2 & t_5^2\\ t_1^4 & t_2^4 & t_3^4 & t_4^4 & t_5^4\\ t_1^8 & t_2^8 & t_3^8 & t_4^8 & t_5^8\\ t_1^{16} & t_2^{16} & t_3^{16} & t_4^{16} & t_5^{16}\end{vmatrix}}$$

$$
\begin{aligned}
c_{5,4} = {} & t_5^{16} + t_4^8 t_5^8 + t_3^4 t_4^4 t_5^8 + t_2^2 t_3^2 t_4^4 t_5^8 + t_1 t_2 t_3^2 t_4^4 t_5^8 + t_1^2 t_3^2 t_4^4 t_5^8 \\
& + t_1 t_2^2 t_3 t_4^4 t_5^8 + t_1^2 t_2 t_3 t_4^4 t_5^8 + t_2^4 t_4^4 t_5^8 + t_1^2 t_2^2 t_4^4 t_5^8 + t_1^4 t_4^4 t_5^8 + t_2^2 t_3^4 t_4^2 t_5^8 \\
& + t_1 t_2 t_3^4 t_4^2 t_5^8 + t_1^2 t_3^4 t_4^2 t_5^8 + t_2^4 t_3^2 t_4^2 t_5^8 + t_1^2 t_2^2 t_3^2 t_4^2 t_5^8 + t_1^4 t_3^2 t_4^2 t_5^8 + t_1 t_2^4 t_3 t_4^2 t_5^8 \\
& + t_1^4 t_2 t_3 t_4^2 t_5^8 + t_1^2 t_2^4 t_4^2 t_5^8 + t_1^4 t_2^2 t_4^2 t_5^8 + t_1 t_2^2 t_3^4 t_4 t_5^8 + t_1^2 t_2 t_3^4 t_4 t_5^8 + t_1 t_2^4 t_3^2 t_4 t_5^8 \\
& + t_1^4 t_2 t_3^2 t_4 t_5^8 + t_1^2 t_2^4 t_3 t_4 t_5^8 + t_1^4 t_2^2 t_3 t_4 t_5^8 + t_3^8 t_5^8 + t_2^4 t_3^4 t_5^8 + t_1^2 t_2^2 t_3^4 t_5^8 + t_1^4 t_3^4 t_5^8 \\
& + t_1^2 t_2^4 t_3^2 t_5^8 + t_1^4 t_2^2 t_3^2 t_5^8 + t_2^8 t_5^8 + t_1^4 t_2^4 t_5^8 + t_1^8 t_5^8 + t_3^4 t_4^8 t_5^4 + t_2^2 t_3^2 t_4^8 t_5^4 \\
& + t_1 t_2 t_3^2 t_4^8 t_5^4 + t_1^2 t_3^2 t_4^8 t_5^4 + t_1 t_2^2 t_3 t_4^8 t_5^4 + t_1^2 t_2 t_3 t_4^8 t_5^4 + t_2^4 t_4^8 t_5^4 + t_1^2 t_2^2 t_4^8 t_5^4 \\
& + t_1^4 t_4^8 t_5^4 + t_3^8 t_4^4 t_5^4 + t_2^4 t_3^4 t_4^4 t_5^4 + t_1^2 t_2^2 t_3^4 t_4^4 t_5^4 + t_1^4 t_3^4 t_4^4 t_5^4 + t_1^2 t_2^4 t_3^2 t_4^4 t_5^4 \\
& + t_1^4 t_2^2 t_3^2 t_4^4 t_5^4 + t_2^8 t_4^4 t_5^4 + t_1^4 t_2^4 t_4^4 t_5^4 + t_1^8 t_4^4 t_5^4 + t_2^2 t_3^8 t_4^2 t_5^4 + t_1 t_2 t_3^8 t_4^2 t_5^4 \\
& + t_1^2 t_3^8 t_4^2 t_5^4 + t_1^2 t_2^4 t_3^4 t_4^2 t_5^4 + t_1^4 t_2^2 t_3^4 t_4^2 t_5^4 + t_2^8 t_3^2 t_4^2 t_5^4 + t_1^4 t_2^4 t_3^2 t_4^2 t_5^4 + t_1^8 t_3^2 t_4^2 t_5^4 \\
& + t_1 t_2^8 t_3 t_4^2 t_5^4 + t_1^8 t_2 t_3 t_4^2 t_5^4 + t_1^2 t_2^8 t_4^2 t_5^4 + t_1^8 t_2^2 t_4^2 t_5^4 + t_1 t_2^2 t_3^8 t_4 t_5^4 + t_1^2 t_2 t_3^8 t_4 t_5^4 \\
& + t_1 t_2^8 t_3^2 t_4 t_5^4 + t_1^8 t_2 t_3^2 t_4 t_5^4 + t_1^2 t_2^8 t_3 t_4 t_5^4 + t_1^8 t_2^2 t_3 t_4 t_5^4 + t_2^4 t_3^8 t_5^4 + t_1^2 t_2^2 t_3^8 t_5^4 \\
& + t_1^4 t_3^8 t_5^4 + t_2^8 t_3^4 t_5^4 + t_1^4 t_2^4 t_3^4 t_5^4 + t_1^8 t_3^4 t_5^4 + t_1^2 t_2^8 t_3^2 t_5^4 + t_1^8 t_2^2 t_3^2 t_5^4 + t_1^4 t_2^8 t_5^4 + t_1^8 t_2^4 t_5^4 \\
& + t_2^2 t_3^4 t_4^8 t_5^2 + t_1 t_2 t_3^4 t_4^8 t_5^2 + t_1^2 t_3^4 t_4^8 t_5^2 + t_2^4 t_3^2 t_4^8 t_5^2 + t_1^2 t_2^2 t_3^2 t_4^8 t_5^2 + t_1^4 t_3^2 t_4^8 t_5^2 \\
& + t_1 t_2^4 t_3 t_4^8 t_5^2 + t_1^4 t_2 t_3 t_4^8 t_5^2 + t_1^2 t_2^4 t_4^8 t_5^2 + t_1^4 t_2^2 t_4^8 t_5^2 + t_2^2 t_3^8 t_4^4 t_5^2 + t_1 t_2 t_3^8 t_4^4 t_5^2 \\
& + t_1^2 t_3^8 t_4^4 t_5^2 + t_1^2 t_2^4 t_3^4 t_4^4 t_5^2 + t_1^4 t_2^2 t_3^4 t_4^4 t_5^2 + t_2^8 t_3^2 t_4^4 t_5^2 + t_1^4 t_2^4 t_3^2 t_4^4 t_5^2 + t_1^8 t_3^2 t_4^4 t_5^2 \\
& + t_1 t_2^8 t_3 t_4^4 t_5^2 + t_1^8 t_2 t_3 t_4^4 t_5^2 + t_1^2 t_2^8 t_4^4 t_5^2 + t_1^8 t_2^2 t_4^4 t_5^2 + t_2^4 t_3^8 t_4^2 t_5^2 + t_1^2 t_2^2 t_3^8 t_4^2 t_5^2 \\
& + t_1^4 t_3^8 t_4^2 t_5^2 + t_2^8 t_3^4 t_4^2 t_5^2 + t_1^4 t_2^4 t_3^4 t_4^2 t_5^2 + t_1^8 t_3^4 t_4^2 t_5^2 + t_1^2 t_2^8 t_3^2 t_4^2 t_5^2 + t_1^8 t_2^2 t_3^2 t_4^2 t_5^2 \\
& + t_1^4 t_2^8 t_4^2 t_5^2 + t_1^8 t_2^4 t_4^2 t_5^2 + t_1 t_2^4 t_3^8 t_4 t_5^2 + t_1^4 t_2 t_3^8 t_4 t_5^2 + t_1 t_2^8 t_3^4 t_4 t_5^2 + t_1^8 t_2 t_3^4 t_4 t_5^2 \\
& + t_1^4 t_2^8 t_3 t_4 t_5^2 + t_1^8 t_2^4 t_3 t_4 t_5^2 + t_1^2 t_2^4 t_3^8 t_5^2 + t_1^4 t_2^2 t_3^8 t_5^2 + t_1^2 t_2^8 t_3^4 t_5^2 + t_1^8 t_2^2 t_3^4 t_5^2 \\
& + t_1^4 t_2^8 t_3^2 t_5^2 + t_1^8 t_2^4 t_3^2 t_5^2 + t_1 t_2^2 t_3^4 t_4^8 t_5 + t_1^2 t_2 t_3^4 t_4^8 t_5 + t_1 t_2^4 t_3^2 t_4^8 t_5 + t_1^4 t_2 t_3^2 t_4^8 t_5 \\
& + t_1^2 t_2^4 t_3 t_4^8 t_5 + t_1^4 t_2^2 t_3 t_4^8 t_5 + t_1 t_2^2 t_3^8 t_4^4 t_5 + t_1^2 t_2 t_3^8 t_4^4 t_5 + t_1 t_2^8 t_3^2 t_4^4 t_5 + t_1^8 t_2 t_3^2 t_4^4 t_5 \\
& + t_1^2 t_2^8 t_3 t_4^4 t_5 + t_1^8 t_2^2 t_3 t_4^4 t_5 + t_1 t_2^4 t_3^8 t_4^2 t_5 + t_1^4 t_2 t_3^8 t_4^2 t_5 + t_1 t_2^8 t_3^4 t_4^2 t_5 + t_1^8 t_2 t_3^4 t_4^2 t_5 \\
& + t_1^4 t_2^8 t_3 t_4^2 t_5 + t_1^8 t_2^4 t_3 t_4^2 t_5 + t_1^2 t_2^4 t_3^8 t_4 t_5 + t_1^4 t_2^2 t_3^8 t_4 t_5 + t_1^2 t_2^8 t_3^4 t_4 t_5 + t_1^8 t_2^2 t_3^4 t_4 t_5 \\
& + t_1^4 t_2^8 t_3^2 t_4 t_5 + t_1^8 t_2^4 t_3^2 t_4 t_5 + t_4^{16} + t_3^8 t_4^8 + t_2^4 t_3^4 t_4^8 + t_1^2 t_2^2 t_3^4 t_4^8 + t_1^4 t_3^4 t_4^8 + t_1^2 t_2^4 t_3^2 t_4^8 \\
& + t_1^4 t_2^2 t_3^2 t_4^8 + t_2^8 t_4^8 + t_1^4 t_2^4 t_4^8 + t_1^8 t_4^8 + t_2^4 t_3^8 t_4^4 + t_1^2 t_2^2 t_3^8 t_4^4 + t_1^4 t_3^8 t_4^4 + t_2^8 t_3^4 t_4^4 \\
& + t_1^4 t_2^4 t_3^4 t_4^4 + t_1^8 t_3^4 t_4^4 + t_1^2 t_2^8 t_3^2 t_4^4 + t_1^8 t_2^2 t_3^2 t_4^4 + t_1^4 t_2^8 t_4^4 + t_1^8 t_2^4 t_4^4 + t_1^2 t_2^4 t_3^8 t_4^2 \\
& + t_1^4 t_2^2 t_3^8 t_4^2 + t_1^2 t_2^8 t_3^4 t_4^2 + t_1^8 t_2^2 t_3^4 t_4^2 + t_1^4 t_2^8 t_3^2 t_4^2 + t_1^8 t_2^4 t_3^2 t_4^2 + t_3^{16} + t_2^8 t_3^8 + t_1^4 t_2^4 t_3^8 \\
& + t_1^8 t_3^8 + t_1^4 t_2^8 t_3^4 + t_1^8 t_2^4 t_3^4 + t_2^{16} + t_1^8 t_2^8 + t_1^{16}
\end{aligned}
$$

Finally, we show the action of some cohomology operations on the generators, [11, Ch. 10].

Proposition 1.1 **(a)** *$Sq^4(x_8) = x_{12}$, $Sq^4(x_{12}) = 0$, $Sq^4(x_{16}) = x_{20}$ and $Sq^8(x_{12}) = x_{20} + x_8 x_{12}$.*

(b) *$Sq^2(y_4) = y_6$, $Sq^2(y_6) = 0$, $Sq^2(y_8) = y_{10}$ and $Sq^4(y_6) = y_4 y_6 + y_{10}$.*

(c) *$Sq^i(y_{32}) = 0$ for $1 \leq i \leq 14$. Put $a = y_4^4 + y_6 y_{10} + y_8^2$, then $Sq^{16}(y_{32}) = a y_{32}$, $Sq^{18}(y_{32}) = Sq^{20}(y_{32}) = Sq^{22}(y_{32}) = Sq^{26}(y_{32}) = 0$, $Sq^{24}(y_{32}) = Sq^8(a) y_{32}$, $Sq^{28}(y_{32}) = Sq^{12}(a) y_{32}$, $Sq^{30}(y_{32}) = Sq^{14}(a) y_{32}$ and $Sq^{32}(y_{32}) = y_{32}^2$.*

Proof We will show only $Sq^4(x_8) = x_{12}$, since the rest of our proof can be very similar.

We note the following:

$$x_8 = \underset{(1)}{\underline{t_5^4}} + \underset{(2)}{\underline{c_2 t_5^2}} + \underset{(3)}{\underline{(c_1c_2 + c_3)t_5}} + \underset{(4)}{\underline{c_2^2}}$$

$$x_{12} = \underset{(2)}{\underline{c_2t_5^4 + c_2^2t_5^2}} + \underset{(3)}{\underline{(c_1^3c_2 + c_1^2c_3 + c_1c_2^2 + c_1c_4 + c_2c_3)t_5}} + \underset{(4)}{\underline{(c_1c_2 + c_3)^2}}$$

Recall the Wu formula :

$$Sq^2(c_2) = c_1c_2 + c_3,$$

$$Sq^2(c_3) = c_1c_3,$$

$$Sq^4(c_3) = c_1c_4 + c_2c_3$$

Each of the calculations for the corresponding parts is as follows:

(1)

$$Sq^4(t_5^4) = Sq^2(t_5^2) \cdot Sq^2(t_5^2) = 0$$

(2)

$$\begin{aligned} Sq^4(c_2t_5^2) &= Sq^4(c_2) \cdot t_5^2 + Sq^2(c_2) \cdot Sq^2(t_5^2) + c_2 \cdot Sq^4(t_5^2) \\ &= c_2^2 \cdot t_5^2 + (c_1c_2 + c_3) \cdot 0 + c_2 \cdot t_5^4 \\ &= c_2t_5^4 + c_2^2t_5^2 \end{aligned}$$

(3)

$$\begin{aligned} Sq^4((c_1c_2 + c_3)t_5) &= Sq^2(c_1c_2 + c_3) \cdot Sq^2(t_5) + Sq^4(c_1c_2 + c_3) \cdot t_5 \\ &= (Sq^2c_1 \cdot c_2 + c_1Sq^2c_2 + Sq^2(c_3))t_5 + (Sq^2c_1 \cdot Sq^2c_2 + c_1Sq^4c_2 + Sq^4(c_3))t_5 \\ &= (c_1^2c_2 + c_1(c_1c_2 + c_3) + c_1c_3)t_5 + (c_1^2(c_1c_2 + c_3) + c_1c_2^2 + c_1c_4 + c_2c_3)t_5 \\ &= (c_1^3c_2 + c_1^2c_3 + c_1c_2^2 + c_1c_4 + c_2c_3)t_5 \end{aligned}$$

(4)

$$\begin{aligned} Sq^4(c_2^2) &= (Sq^2c_2)^2 \\ &= (c_1c_2 + c_3)^2 \end{aligned}$$

This completes the proof.

$\square$

2 Invariant rings over the rational numbers

The representation of $W(SU(3)) \cong \Sigma_3$ is generated by the 2 reflections

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

For $PU(n)$, its Weyl group can be

$$W(PU(3)) = \phi_3 W(SU(n)) \phi_3^{-1} \quad \text{for} \quad \phi_3 = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

so that the corresponding matrices are

$$\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & -3 \\ 0 & 1 \end{pmatrix}$$

(see [2, 3]).

Let $S(V) = \mathbb{Z}[t_1, t_2]$ and $W = W(PU(3))$, and put $s_1 = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and $s_2 = \begin{pmatrix} -1 & -3 \\ 0 & 1 \end{pmatrix}$. If

$$\alpha = 3t_1^2 - 3t_1t_2 + t_2^2 \quad \text{and} \quad \beta = t_2(3t_1 - t_2)(3t_1 - 2t_2) = 9t_1^2t_2 - 9t_1t_2^2 + 2t_2^3,$$

then we see that

$$s_1\alpha = 3(2t_1 - t_2)^2 - 3(2t_1 - t_2)(3t_1 - 2t_2) + (3t_1 - 2t_2)^2 = \alpha,$$

$s_2\alpha = 3(-t_1)^2 - 3(-t_1)(-3t_1 + t_2) + (-3t_1 + t_2)^2 = \alpha$, $s_1\beta = \beta$ and $s_2\beta = \beta$, and hence $\alpha,\ \beta \in S(V)^W$. By [11, Proposition 5.3.7], $\{\alpha, \beta\}$ is a system of parameters since the system of equations $3t_1^2 - 3t_1t_2 + t_2^2 = 0$ and $9t_1^2t_2 - 9t_1t_2^2 + 2t_2^3 = 0$ has only trivial solution $t_1 = t_2 = 0$. By [11, Proposition 5.5.5], over the field $\mathbb{Q}$, we obtain

$$S(V)^W \otimes \mathbb{Q} = \mathbb{Q}[\alpha, \beta]$$

since $\deg(\alpha)\deg(\beta) = 2 \cdot 3 = 6 = |W|$. We have

$$\begin{array}{ccc} S(V)^W & \supset & \mathbb{Z}[\alpha, \beta] \\ \cap & & \cap \\ S(V)^W \otimes \mathbb{Q} & = & \mathbb{Q}[\alpha, \beta]. \end{array}$$

We see, however, that

$$\mathbb{Q}[\alpha, \beta] \cap S(V)^W \neq \mathbb{Z}[\alpha, \beta].$$

Here we claim that, if

$$P(t_1, t_2) = 9t_1^6 - 27t_1^5t_2 + 90t_1^4t_2^2 - 135t_1^3t_2^3 + 90t_1^2t_2^4 - 27t_1t_2^5 + 3t_2^6,$$

then

$$P(t_1, t_2) = \frac{1}{3}\alpha^3 + \frac{2}{3}\beta^2 \notin \mathbb{Z}[\alpha, \beta].$$

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