Matrices and mod p admissible maps for classifying spaces

Kenshi Ishiguro[1], Takahiro Koba[2], Yusuke Ueno[3] and Fumihisa Yayama[4]

(*Received May 31, 2017*)

Abstract

An admissible map for classifying spaces can be regarded as a matrix. We discuss the diagonalizability of such matrices as well as the Jordan canonical forms. Sometimes a high–dimensional behavior characterizes the induced homomorphism of the cohomology. We will ask if such a thing happens in our case. A relationship between the diagonalizability of admissible maps and the reducibility of classifying spaces is also discussed for unitary groups.

AMS Classification 55R35; 15A21, 55P60

Keywords classifying space, Lie group, mod p cohomology, Jordan canonical form, admissible map

We first recall admissible maps for the rational cohomology, [1]. As explained in [8], it is well–known that, for a connected compact Lie group G, the rational cohomology $H^*(BG,\mathbb{Q})$ is isomorphic to the ring of invariants under the action of the Weyl group $W(G)$. Consequently, for connected compact Lie groups G and K with maximal tori T_G and T_K respectively, we see $H^*(BG;\mathbb{Q}) \cong H^*(BT_G;\mathbb{Q})^{W(G)}$ and $H^*(BK;\mathbb{Q}) \cong H^*(BT_K;\mathbb{Q})^{W(K)}$. For any map $f : BG \longrightarrow BK$ we have the commutative diagram:

$$\begin{array}{ccc} H^*(BT_K;\mathbb{Q}) & \xrightarrow{\phi_f} & H^*(BT_G;\mathbb{Q}) \\ \uparrow & & \uparrow \\ H^*(BK;\mathbb{Q}) & \xrightarrow[f^*]{} & H^*(BG;\mathbb{Q}) \end{array}$$

Here $\phi = \phi_f$ is *admissible* ; namely for any $w \in W(G)$ we can find $w' \in W(K)$ such that $w\phi = \phi w'$.

Recall next that $H^*(BT^n;\mathbb{Q}) = \mathbb{Q}[t_1, t_2, \cdots, t_n]$ is a polynomial ring in n variables of degree 2. So the admissible map ϕ can be regarded as a $rank(G) \times$

[1]Department of Applied Mathematics, Fukuoka Univ., Fukuoka, 814-0180, Japan
[2]Goshi junior high school, Kumamoto, 869-0180, Japan
[3]Department of Applied Mathematics, Fukuoka Univ., Fukuoka, 814-0180, Japan
[4]Nishijoyo senior high school, Kyoto, 610-0117, Japan

$rank(K)$ matrix, since the ring homomorphism is determined by a linear map on the vector space $H^2(BT_K;\mathbb{Q})$. For instance, the admissible self–maps for $H^*(BU(n);\mathbb{F}_p) \cong H^*(BT^n;\mathbb{F}_p)^{\Sigma_n}$ are as follows:

$$\phi = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \text{ or } \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ & & \ddots & \\ b & b & \cdots & a \end{pmatrix}$$

Notice, [2], that the rational cohomology can be replaced by the mod p cohomology when p is large. We note that $H^*(BG;\mathbb{F}_p)$ is isomorphic to $H^*(BT_G;\mathbb{F}_p)^{W(G)}$, for instance, if p does not divide the order of $W(G)$. Any map from $H^*(BT^n;\mathbb{F}_p)$ to $H^*(BT^m;\mathbb{F}_p)$ over the Steenrod algebra $\mathcal{A}_p$ can be determined by a matrix. Conversely, any matrix gives such an $\mathcal{A}_p$–map.

In §1, for an admissible map $\phi : H^*(BT^n;\mathbb{F}_p)\longrightarrow H^*(BT^n;\mathbb{F}_p)$, we consider the case that the $n\times n$ matrix $\phi_2 : H^2(BT^n;\mathbb{F}_p)\longrightarrow H^2(BT^n;\mathbb{F}_p)$ is diagonalizable. We will show that, in this case, any matrix $\phi_{2k} : H^{2k}(BT^n;\mathbb{F}_p)\longrightarrow H^{2k}(BT^n;\mathbb{F}_p)$ is also diagonalizable, and that if ϕ_2 is invertible, so is ϕ_{2k} as a consequence. In §2, we consider the case that ϕ_2 is not diagonalizable. Even though ϕ_2 is in a Jordan canonical form, ϕ_{2k} need not be so. Therefore we will find an invertible matrix P_{2k} such that the conjugate $P_{2k}^{-1}\phi_{2k}P_{2k}$ is a Jordan canonical form. The authors announced some results related to this work at regional meetings of the Japan Math. Soc., [11], [6] and [7]. A high–dimensional behavior is discussed in §3. We consider a converse to Lemma 1.1 and see that it doesn't seem to be the case in a narrow sense. Finally in §4, the diagonalizability of admissible maps and the reducibility of classifying spaces are considered. It turns out that, for unitary groups, both phenomena happen under the same condition.

1 Diagonalizable Matrices

An admissible map $\phi : H^*(BT^n;\mathbb{F}_p)\longrightarrow H^*(BT^n;\mathbb{F}_p)$ can be regarded as a square matrix on each $H^{2k}(BT^n;\mathbb{F}_p)$. Let A denote the matrix presentation of ϕ_2, and let $\rho_k(A)$ denote the matrix presentation of ϕ_{2k}. For $n = 2$, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\rho_2(A) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$. Because if $\phi_2(t_1) = at_1 + bt_2$, then $\phi_4(t_1^2) = a^2t_1^2 + 2abt_1t_2 + b^2t_2^2$ and so on. We note that if the matrix A is diagonal, so is $\rho_k(A)$ for all $k \geq 1$.

Lemma 1.1 *If the matrix presentation of ϕ_2 on $H^2(BT^n;\mathbb{F}_p)$ is diagonalizable, so is that of ϕ_{2k} for all $k \geq 1$.*

Proof For the matrix presentation A of ϕ_2, we see the following:

$\rho_k(PAP^{-1}) = \rho_k(P)\rho_k(A)\rho_k(P^{-1})$.

$$\begin{array}{ccc} H^*(BT^n;\mathbb{F}_p) & \xrightarrow{A} & H^*(BT^n;\mathbb{F}_p) \\ {\scriptstyle P^{-1}}\uparrow & & \downarrow{\scriptstyle P} \\ H^*(BT^n;\mathbb{F}_p) & \xrightarrow[PAP^{-1}]{} & H^*(BT^n;\mathbb{F}_p) \end{array}$$

If A is diagonalizable, there is an invertible matrix P such that PAP^{-1} is a diagonal matrix. Thus $\rho_k(PAP^{-1})$ is a diagonal matrix. □

There are admissible maps that are not diagonalizable. For instance, let $A = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$ and $\phi_2 = \rho_1(A)$. For the exceptional Lie group G_2, the map ϕ is admissible, [8, Proposition 2].

$$\begin{array}{ccc} H^*(BT^2;\mathbb{Q}) & \xrightarrow{\phi} & H^*(BT^2;\mathbb{Q}) \\ \uparrow & & \uparrow \\ H^*(BG_2;\mathbb{Q}) & \longrightarrow & H^*(BG_2;\mathbb{Q}) \end{array}$$

Since $A^2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, we see that $\phi^2 = 0$ at $p = 3$. So we consider the map $\phi : H^*(BT^2;\mathbb{F}_3) \longrightarrow H^*(BT^2;\mathbb{F}_3)$. If ϕ_2 were diagonalizable, then $PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for an invertible matrix P. Note that $A^2 = 0$ implies $(PAP^{-1})^2 = 0$. This would mean that $\lambda_1 = \lambda_2 = 0$. This is a contradiction.

Concerning the determinants, one can show the following:

Lemma 1.2 $\det \phi_{2\mathrm{k}} = (\det \phi_2)^{\frac{\mathrm{k}(\mathrm{k}+1)}{2}}$ *for all $k \geq 1$.*

2 Jordan canonical form

Next we consider the maps $\phi : H^*(BT^2;\mathbb{F}_p) \longrightarrow H^*(BT^2;\mathbb{F}_p)$ when ϕ_2 is not diagonalizable. Particularly we will treat the case $\phi_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, which means that $\phi_2(t_1) = \lambda t_1 + t_2$, $\phi_2(t_2) = \lambda t_2$ for $H^*(BT^2;\mathbb{F}_p) = \mathbb{F}_p[t_1, t_2]$.

For instance, we see that $\phi_4 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} {}_2C_2\lambda^2 & {}_2C_1\lambda & {}_2C_0 \\ 0 & {}_1C_1\lambda^2 & {}_1C_0\lambda \\ 0 & 0 & {}_0C_0\lambda^2 \end{pmatrix}$, where nC_k denotes the binomial coefficient, as usual. We note that $P_4^{-1}\phi_4 P_4 = \begin{pmatrix} \lambda^2 & 1 & 0 \\ 0 & \lambda^2 & 1 \\ 0 & 0 & \lambda^2 \end{pmatrix}$ if we take $P_4 = \begin{pmatrix} 1 & \frac{1}{2\lambda^2} & 0 \\ 0 & \frac{1}{2\lambda} & 0 \\ 0 & 0 & \frac{1}{2\lambda^2} \end{pmatrix}$.

Theorem 1 *For the two square matrices of size* $n+1$

$$J_{2n} = \begin{pmatrix} \lambda^n & 1 & \dots & & & 0 \\ & \lambda^n & 1 & \dots & & 0 \\ & & \ddots & \ddots & & 0 \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 1 \\ 0 & \dots & & & & \lambda^n \end{pmatrix}$$

and
$$\phi_{2n} = \begin{pmatrix} {}_nC_n\lambda^n & {}_nC_{n-1}\lambda^{n-1} & \dots & & & {}_nC_0\lambda^1 \\ & {}_{n-1}C_{n-1}\lambda^n & {}_{n-1}C_{n-2}\lambda^{n-1} & \dots & & {}_{n-1}C_0\lambda^2 \\ & & {}_{n-2}C_{n-2}\lambda^n & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & {}_1C_1\lambda^n & {}_1C_0\lambda^{n-1} \\ 0 & & \dots & & & {}_0C_0\lambda^n \end{pmatrix},$$

$P_{2n}^{-1}\phi_{2n}P_{2n} = J_{2n}$ *holds if* P_{2n} *is the following upper triangular matrix:*

$P_{2n} =$

$$\begin{pmatrix} \overbrace{\sum_{i=1}^{n-p} {}_{n-p+1}C_i \lambda^{n-i}(p+i,q+1)}^{1\le q\le n-2} & \overbrace{\sum_{i=1}^{n-p} {}_{n-p+1}C_i \lambda^{n+p-1}\frac{1}{n!\lambda^{n(n-1)}}}^{q=n-1} & \overbrace{\frac{\lambda^0}{n!\lambda^{n(n-1)}}}^{q=n} & \overbrace{0}^{q=n+1} \\ \vdots & \vdots & \frac{\lambda^1}{n!\lambda^{n(n-1)}} & \vdots \\ & & \vdots & \\ & & \frac{\lambda^{n-1}}{n!\lambda^{n(n-1)}} & 0 \\ 0 & 0 & 0 & \frac{1}{n!\lambda^{n(n-1)}} \end{pmatrix}$$

Here (p,q) means the (p,q)–entry of P_{2n}. The left column means the type of the first $n-2$ columns so that each entry varies upon q for $1 \le q \le n-2$. For instance, P_{10} is given as follows:

$$P_{10} = \begin{pmatrix} \frac{120\lambda^{20}}{120\lambda^{20}} & \frac{240\lambda^{15}}{120\lambda^{20}} & \frac{150\lambda^{10}}{120\lambda^{20}} & \frac{30\lambda^{5}}{120\lambda^{20}} & \frac{1\lambda^{0}}{120\lambda^{20}} & 0 \\ 0 & \frac{24\lambda^{16}}{120\lambda^{20}} & \frac{36\lambda^{11}}{120\lambda^{20}} & \frac{14\lambda^{6}}{120\lambda^{20}} & \frac{1\lambda^{1}}{120\lambda^{20}} & 0 \\ 0 & 0 & \frac{6\lambda^{12}}{120\lambda^{20}} & \frac{6\lambda^{7}}{120\lambda^{20}} & \frac{1\lambda^{2}}{120\lambda^{20}} & 0 \\ 0 & 0 & 0 & \frac{2\lambda^{8}}{120\lambda^{20}} & \frac{1\lambda^{3}}{120\lambda^{20}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1\lambda^{4}}{120\lambda^{20}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{120\lambda^{20}} \end{pmatrix}$$

Proof It is enough to show $\phi_{2n}P_{2n} = P_{2n}J_{2n}$. We will determine the columns of P_{2n} in the reversed order, since we need the $k+1$-st column to find the k-th column for $k = 1, 2, \cdots, n$.

First consider the $n+1$-st column. Multiplying both sides by $n!\lambda^{n(n-1)}$ we will show the following:

$$\phi_{2n} \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{q=n+1} = n!\lambda^{n(n-1)} P_{2n} \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda^n \end{pmatrix}}_{q=n+1}$$

To do so, we compute both sides as follows:

$$\phi_{2n}\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}=\begin{pmatrix}\lambda^0\\ \lambda^1\\ \vdots\\ \lambda^n\end{pmatrix},\quad \text{and}\quad n!\lambda^{n(n-1)}P_{2n}\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ \lambda^n\end{pmatrix}=\begin{pmatrix}\lambda^0\\ \lambda^1\\ \vdots\\ \lambda^n\end{pmatrix}$$

Consequently we see that the both $n+1$-st columns are the same.

Next consider the n-th column. Again multiplying by $n!\lambda^{n(n-1)}$ we will show the following:

$$\phi_{2n}\underbrace{\begin{pmatrix}\lambda^0\\ \vdots\\ \lambda^{n-1}\\ 0\end{pmatrix}}_{q=n}=n!\lambda^{n(n-1)}P_{2n}\underbrace{\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ \lambda^n\\ 0\end{pmatrix}}_{q=n}$$

To do so once again, we compute both sides as follows:

.

$$\phi_{2n}\begin{pmatrix}\lambda^0\\ \vdots\\ \lambda^{n-1}\\ 0\end{pmatrix}=\begin{pmatrix}{}_nC_n\lambda^n\lambda^0+{}_nC_{n-1}\lambda^{n-1}\lambda^1+\cdots+{}_nC_1\lambda^1\lambda^{n-1}\\ {}_{n-1}C_{n-1}\lambda^n\lambda^1+{}_{n-1}C_{n-2}\lambda^{n-1}\lambda^2+\cdots+{}_{n-1}C_1\lambda^2\lambda^{n-1}\\ \vdots\\ {}_1C_1\lambda^n\lambda^{n-1}\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\lambda^n\sum_{i=1}^{n}{}_nC_i\\ \lambda^{n+1}\sum_{i=1}^{n-1}{}_{n-1}C_i\\ \vdots\\ \lambda^{2n-1}\sum_{i=1}^{1}{}_1C_i\\ 0\end{pmatrix}$$

.

$$n!\lambda^{n(n-1)}P_{2n}\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ \lambda^n\\ 0\end{pmatrix}=\begin{pmatrix}\sum_{i=1}^{n-1}{}_nC_i\lambda^n+\lambda^0\lambda^n\\ \sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n+1}+\lambda^1\lambda^n\\ \vdots\\ 0+\lambda^{n-1}\lambda^n\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\lambda^n\sum_{i=1}^{n}{}_nC_i\\ \lambda^{n+1}\sum_{i=1}^{n-1}{}_{n-1}C_i\\ \vdots\\ \lambda^{2n-1}\sum_{i=1}^{1}{}_1C_i\\ 0\end{pmatrix}$$

Next consider the $n-1$-st column. Again multiplying by $n!\lambda^{n(n-1)}$ we will show the following:

$$\phi_{2n}\underbrace{\begin{pmatrix}\sum_{i=1}^{n-p}{}_{n-p+1}C_i\lambda^{n+p+1}\\ \vdots\\ \sum_{i=1}^{n-p}{}_{n-p+1}C_i\lambda^{n+p+1}\\ 0\\ 0\end{pmatrix}}_{q=n-1}=n!\lambda^{n(n-1)}P_{2n}\underbrace{\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ \lambda^n\\ 0\\ 0\end{pmatrix}}_{q=n-1}$$

To do so, we compute both sides as follows:

$$\phi_{2n}\begin{pmatrix}\sum_{i=1}^{n-p}{}_{n-p+1}C_i\lambda^{n+p+1}\\ \vdots\\ \sum_{i=1}^{n-p}{}_{n-p+1}C_i\lambda^{n+p+1}\\ 0\\ 0\end{pmatrix}$$

$$= \begin{pmatrix} ({}_nC_n\lambda^n\sum_{i=1}^{n-1}{}_nC_i\lambda^n) + ({}_nC_{n-1}\lambda^{n-1}\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n+1}) + ({}_nC_{n-2}\lambda^{n-2}\sum_{i=1}^{n-3}{}_{n-2}C_i\lambda^{n+2}) + \\ \cdots + ({}_nC_2\lambda^2\sum_{i=1}^{n-(n-1)}{}_2C_i\lambda^{n+(n-2)}) + ({}_nC_1\lambda^1\cdot 0) + ({}_nC_0\lambda^0\cdot 0) \\ ({}_{n-1}C_{n-1}\lambda^n\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n+1}) + ({}_{n-1}C_{n-2}\lambda^{n-1}\sum_{i=1}^{n-3}{}_{n-2}C_i\lambda^{n+2}) + ({}_{n-1}C_{n-3}\lambda^{n-2}\sum_{i=1}^{n-4}{}_{n-3}C_i\lambda^{n+3}) + \\ \cdots + ({}_{n-1}C_2\lambda^3\sum_{i=1}^{n-(n-1)}{}_2C_i\lambda^{2n-2}) + ({}_{n-1}C_1\lambda^2\cdot 0) + ({}_{n-1}C_0\lambda^1\cdot 0) \\ \vdots \\ ({}_3C_3\lambda^n\sum_{i=1}^{n-(n-2)}{}_3C_i\lambda^{2n-3}) + ({}_3C_2\lambda^{n-1}\sum_{i=1}^{n-(n-1)}{}_2C_i\lambda^{2n-2}) + ({}_3C_1\lambda^{n-2}\cdot 0) + ({}_3C_0\lambda^{n-3}\cdot 0) \\ ({}_2C_2\lambda^n\sum_{i=1}^{n-(n-1)}{}_2C_i\lambda^{2n-2}) + ({}_2C_1\lambda^{n-1}\cdot 0) + ({}_2C_0\lambda^{n-2}\cdot 0) \\ ({}_1C_1\lambda^n\cdot 0) + ({}_1C_0\lambda^{n-1}\cdot 0) \\ ({}_0C_0\lambda^n\cdot 0) \end{pmatrix}$$

.

$$n!\lambda^{n(n-1)}P_{2n}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda^n \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \{\sum_{i=1}^{n-1}{}_nC_i\lambda^{n-i}(1+i,n-1)\} + \{\sum_{i=1}^{n-1}{}_nC_i\lambda^n\}\lambda^n \\ \{\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n-i}(2+i,n-1)\} + \{\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n+1}\}\lambda^n \\ \vdots \\ \{\sum_{i=1}^{2}{}_3C_i\lambda^{n-i}(n-2+i,n-1)\} + \{\sum_{i=1}^{2}{}_3C_i\lambda^{2n-3}\}\lambda^n \\ 0 + \{\sum_{i=1}^{1}{}_2C_i\lambda^{2n-2}\}\lambda^n \\ 0 \\ 0 \end{pmatrix}$$

$$
.= \begin{pmatrix}
\{{}_nC_1\lambda^{n-1}(\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n+1})\}+\{{}_nC_2\lambda^{n-2}(\sum_{i=1}^{n-3}{}_{n-2}C_i\lambda^{n+2})\}+\cdots+ \\
\{{}_nC_{n-2}\lambda^2(\sum_{i=1}^{1}{}_2C_i\lambda^{2n-2})\}+\{{}_nC_{n-1}\lambda^1\cdot 0\}+\{\sum_{i=1}^{n-1}{}_nC_i\lambda^n\}\lambda^n \\
\{{}_{n-1}C_1\lambda^{n-1}(\sum_{i=1}^{n-3}{}_{n-2}C_i\lambda^{n+2})\}+\{{}_{n-1}C_2\lambda^{n-2}(\sum_{i=1}^{n-4}{}_{n-3}C_i\lambda^{n+3})\}+\cdots+ \\
\{{}_{n-1}C_{n-3}\lambda^3(\sum_{i=1}^{1}{}_2C_i\lambda^{2n-2})\}+\{{}_{n-1}C_{n-2}\lambda^2\cdot 0\}+\{\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n+1}\}\lambda^n \\
\vdots \\
\{{}_3C_1\lambda^{n-1}(\sum_{i=1}^{1}{}_2C_i\lambda^{2n-2})\}+\{{}_3C_2\lambda^{n-2}\cdot 0\}+\{\sum_{i=1}^{2}{}_3C_i\lambda^{2n-3}\}\lambda^n \\
\{\sum_{i=1}^{1}{}_2C_i\lambda^{2n-2}\}\lambda^n \\
0 \\
0
\end{pmatrix}
$$

Consequently we see that the both $n-1$-st columns are the same, since ${}_nC_a = {}_nC_{n-a}$.

Next we consider the $n-2$-nd column, and show the following:

$$
\phi_{2n}\underbrace{\begin{pmatrix}
\sum_{i=1}^{n-1}{}_nC_i\lambda^{n-i}(1+i,n-1) \\
\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n-i}(2+i,n-1) \\
\vdots \\
0 \\
0
\end{pmatrix}}_{q=n-2} = n!\lambda^{n(n-1)}P_{2n}\underbrace{\begin{pmatrix}
0 \\ \vdots \\ 0 \\ 1 \\ \lambda^n \\ 0 \\ 0 \\ 0
\end{pmatrix}}_{q=n-2}
$$

To do so, we compute the both sides as follows. At the last step, for $2 \le j \le n-2$, the entries $(j, n-2)$ will be replaced by sums of $(j+i, n-1)'s$.

$$
\phi_{2n}\begin{pmatrix}
\sum_{i=1}^{n-1}{}_nC_i\lambda^{n-i}(1+i,n-1) \\
\sum_{i=1}^{n-2}{}_{n-1}C_i\lambda^{n-i}(2+i,n-1) \\
\vdots \\
0 \\
0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\{{}_nC_n\lambda^n \sum_{i=1}^{n-1} {}_nC_i\lambda^{n-i}(1+i,n-1)\} + \{{}_nC_{n-1}\lambda^{n-1} \sum_{i=1}^{n-2} {}_{n-1}C_i\lambda^{n-i}(2+i,n-1)\} + \cdots \\
\cdots + \{{}_nC_3\lambda^3 \sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\} + 0 + 0 + 0 \\
\{{}_{n-1}C_{n-1}\lambda^n \sum_{i=1}^{n-2} {}_{n-1}C_i\lambda^{n-i}(2+i,n-1)\} + \{{}_{n-1}C_{n-2}\lambda^{n-1} \sum_{i=1}^{n-3} {}_{n-2}C_i\lambda^{n-i}(3+i,n-1)\} + \cdots \\
\cdots + \{{}_{n-1}C_3\lambda^4 \sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\} + 0 + 0 + 0 \\
\vdots \\
\{{}_4C_4\lambda^n \sum_{i=1}^{3} {}_4C_i\lambda^{n-i}(n-3+i,n-1)\} + \{{}_4C_3\lambda^{n-1} \sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\} \\
+0+0+0 \\
\{{}_3C_3\lambda^n \sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\} + 0 + 0 + 0 \\
0 \\
0 \\
0
\end{pmatrix}
$$

$$
n!\lambda^{n(n-1)}P_{2n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda^n \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix}
\{\sum_{i=1}^{n-1} {}_nC_i\lambda^{n-i}(1+i,n-2)\} + \{\sum_{i=1}^{n-1} {}_nC_i\lambda^{n-i}(1+i,n-1)\}\lambda^n \\
\{\sum_{i=1}^{n-2} {}_{n-1}C_i\lambda^{n-i}(2+i,n-2)\} + \{\sum_{i=1}^{n-2} {}_{n-1}C_i\lambda^{n-i}(2+i,n-1)\}\lambda^n \\
\vdots \\
\{\sum_{i=1}^{3} {}_4C_i\lambda^{n-i}(n-3+i,n-2)\} + \{\sum_{i=1}^{3} {}_4C_i\lambda^{n-i}(n-3+i,n-1)\}\lambda^n \\
0 + \{\sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\}\lambda^n \\
0 \\
0 \\
0
\end{pmatrix}
$$

$$= \begin{pmatrix} \{{}_nC_1\lambda^{n-1}(2,n-2)\} + \{{}_nC_2\lambda^{n-2}(3,n-2)\} + \cdots + \{{}_nC_{n-3}\lambda^3(n-2,n-2)\} \\ +0+0+\{\sum_{i=1}^{n-1} {}_nC_i\lambda^{n-i}(1+i,n-1)\}\lambda^n \\ \{{}_{n-1}C_1\lambda^{n-1}(3,n-2)\} + \{{}_{n-1}C_2\lambda^{n-2}(4,n-2)\} + \cdots + \{{}_{n-1}C_{n-4}\lambda^4(n-2,n-2)\} \\ +0+0+\{\sum_{i=1}^{n-2} {}_{n-1}C_i\lambda^{n-i}(2+i,n-1)\}\lambda^n \\ \vdots \\ \{{}_4C_1\lambda^{n-1}(n-2,n-2)\}+0+0+\{\sum_{i=1}^{3} {}_4C_i\lambda^{n-i}(n-3+i,n-1)\}\lambda^n \\ 0+0+\{\sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\}\lambda^n \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \{{}_nC_1\lambda^{n-1}\sum_{i=1}^{n-2} {}_{n-1}C_i\lambda^{n-i}(2+i,n-1)\} + \{{}_nC_2\lambda^{n-2}\sum_{i=1}^{n-3} {}_{n-2}C_i\lambda^{n-i}(3+i,n-1)\} + \cdots \\ \cdots + \{{}_nC_{n-3}\lambda^3\sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\}+0+0+\{\sum_{i=1}^{n-1} {}_nC_i\lambda^{n-i}(1+i,n-1)\}\lambda^n \\ \{{}_{n-1}C_1\lambda^{n-1}\sum_{i=1}^{n-3} {}_{n-2}C_i\lambda^{n-i}(3+i,n-1)\} + \{{}_{n-1}C_2\lambda^{n-2}\sum_{i=1}^{n-4} {}_{n-3}C_i\lambda^{n-i}(4+i,n-1)\} + \cdots \\ \cdots + \{{}_{n-1}C_{n-4}\lambda^4\sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\}+0+0+\{\sum_{i=1}^{n-2} {}_{n-1}C_i\lambda^{n-i}(2+i,n-1)\}\lambda^n \\ \vdots \\ \{{}_4C_1\lambda^{n-1}\sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\}+0+0+\{\sum_{i=1}^{3} {}_4C_i\lambda^{n-i}(n-3+i,n-1)\}\lambda^n \\ 0+0+\{\sum_{i=1}^{2} {}_3C_i\lambda^{n-i}(n-2+i,n-1)\}\lambda^n \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Consequently we see that the both are the same. And a similar argument goes on for each step. One can complete the proof.

□

3 High–dimennsional behavior

It is well-known that classifying spaces BG have rigid structure so that there are relatively few maps between them. Sometimes the maps are controlled by mod p cohomology. The fusion version of the rigidity of BG can be found in recent work of [4] and [3].

A result of Mislin [10] states that for a homomorphsim $\rho : G \longrightarrow K$, if there is a non–negative integer n such that $(B\rho)^* : H^j(BK;\mathbb{Z}) \longrightarrow H^j(BG;\mathbb{Z})$ is an isomorphism for all $j \geq n$, then $(B\rho)^*$ is an isomorphism for all $j \geq 0$. This means that a high–dimensional behavior characterizes the induced homomorphsim of the cohomology. Here we ask if $\phi_{2j} : H^{2j}(BT^2;\mathbb{F}_p)\longrightarrow H^{2j}(BT^2;\mathbb{F}_p)$ is diagonalizable, is $\phi_{2(j-1)}$ also diagonalizable ? This is a converse to Lemma 1.1. We will see that the answer turns out to be negative.

First note that if $A \in GL(n,\mathbb{F}_p)$ is diagonalizable, then A^{p-1} must be the identity matrix. Thus any invertible matrix whose order exceeds $p-1$ can not be diagonalizable. Here we consider anti–diagonal matrices. We denote an $n\times n$ anti–diagonal matrix by $\text{anti-d}(\alpha_1,\alpha_2,\cdots,\alpha_{\rm n})$. For instance we see $\begin{pmatrix} 0 & \alpha_2 \\ \alpha_1 & 0 \end{pmatrix} = \text{anti-d}(\alpha_1,\alpha_2)$. If an admissible map $\phi_2 = \text{anti-d}(c,b) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, then $\phi_{2n} = \text{anti-d}(c^n, c^{n-1}b,\cdots,b^n)$. The product of two anti–diagonal matrices is a diagonal matrix. And we denote an $n\times n$ diagonal matrix by $\text{d}(\beta_1,\beta_2,\cdots,\beta_{\rm n})$ so that $\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} = \text{d}(\beta_1,\beta_2)$. Suppose $B = \text{anti-d}(\alpha_1,\alpha_2,\cdots,\alpha_{\rm n})$. Then $B^2 = \text{d}(\alpha_{\rm n}\alpha_1,\alpha_{\rm n-1}\alpha_2,\cdots,\alpha_1\alpha_{\rm n})$.

The unit group of $\mathbb{F}_p$ is isomorphic to the cyclic group $\mathbb{Z}/(p-1)$. Let ζ be a generator of $\mathbb{Z}/(p-1)$. If $\phi_2 = \text{anti-d}(\zeta,1)$, then $\phi_2^2 = \text{d}(\zeta,\zeta)$. Hence the order of ϕ_2 is $2(p-1)$. This means that ϕ_2 is not diagonalizable.

Proposition 3.1 *Suppose* $\phi_2 = \text{anti-d}(\zeta,1)$*. Then we have the following:*

(1) ϕ_4 *is diagonalizable.*

(2) *If* $p-1$ *is a power of* 2*, then* ϕ_{4n-2} *is not diagonalizable for all* $n\geq 1$*.*

Proof (1) First we note that $A = \phi_4 = \text{anti-d}(\zeta^2,\zeta,1)$. If $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \zeta & 0 & -\zeta \end{pmatrix}$, then $P^{-1}AP = \text{d}(\zeta,\zeta,-\zeta)$.

(2) Next we note that $\phi_{2m}^2 = \text{d}(\zeta^m,\zeta^m,\cdots,\zeta^m)$, and hence $\phi_{4n-2}^2 = \text{d}(\zeta^{2n-1},\zeta^{2n-1},\cdots,\zeta^{2n-1})$. Since $p-1$ is a power of 2, the odd number $2n-1$ is prime to $p-1$. Consequently the order of ζ^{2n-1} is $p-1$ so that the order of ϕ_{4n-2} is $2(p-1)$. Therefore ϕ_{4n-2} can not be diagonalizable for all $n\geq 1$ □

When $p=3$ and $\phi_2 = \text{anti-d}(-1,1)$, we see that $\phi_{4n} = \text{anti-d}(1,-1,\cdots,-1,1)$ is a square matrix of size $2n+1$. It is easy to show that each ϕ_{4n} is diagonalizable.

4 The diagonalizability of admissible maps and the reducibility of classifying spaces

We consider if the admissible self–maps for $H^*(BU(n);\mathbb{F}_p) \cong H^*(BT^n;\mathbb{F}_p)^{\Sigma_n}$ are diagonalizable at p:

$$\phi = \begin{pmatrix} a & b & \cdots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix}$$

Proposition 4.1 *The above $n \times n$ matrix ϕ with $b \neq 0$ is diagonalizable at p if and only if n is not divisible by p.*

Proof Let $P = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 1 \\ -1 & \cdots & \cdots & -1 & 1 \end{pmatrix}.$

Since det $P = n$, this matrix P is invertible if n is not divisible by p. One can see that $P^{-1}\phi P$ is the diagonal matrix $\mathrm{d(a-b,\cdots,a-b,a+(n-1)b)}$.

Conversely, assume that n is divisible by p. If ϕ was diagonalizable, its conjugate would be the diagonal matrix $\mathrm{d(a-b,\cdots,a-b,a+(n-1)b)}$. Since n is divisible by p, this diagonal matrix is a scalar matrix. If a conjugate of ϕ is a scalar matrix, so is the matrix ϕ. This contradiction completes the proof. □

The exact sequence $\mathbb{Z}/n \longrightarrow SU(n) \times S^1 \longrightarrow U(n)$ induces a fibration of classifying spaces $B\mathbb{Z}/n \longrightarrow BSU(n) \times BS^1 \longrightarrow BU(n)$. According to a result of [5], $BSU(n) \times BS^1$ is p–equivalent to $BU(n)$ if and only if n is not divisible by p. In this case, any self–map of $BU(n)$ lifts to a self–map of $BSU(n) \times BS^1$ at p. It is known [9] that the admissible map of any self–map of $BSU(n)$ is a scalar matrix. Thus the admissible map for $BU(n)$ must be diagonalizable.

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