Necessary and sufficient degrees of similarity of voters' preferences to deviate from dictatorship

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1 Introduction

A decision based on agents' preferences is called a social choice. Social choices are essential parts of our society, and it is important to make social choices in "nice" ways.

Intuitively, when the agents have diverse preferences, it is difficult to make a social choice, and when the agents have similar preferences, it is easy to make a social choice.[1] Then, what is the necessary degree of similarity of preferences to construct a "nice" way of making social choices? This is the motivating question of this article, and we find an answer.

A rule of choosing one alternative based on agents' preferences is called a *social choice function*. This article investigates the possibility of constructing a "nice" social choice function in the environments such that

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[1] At the extreme case, when all agents have the same preference, then the common top ranked alternative is the only natural candidate for a social choice. On the other hand, when agents may have diverse preferences, it is known that constructing a "nice" social choice rule is difficult, and sometimes impossible.

- there is the preference called the "center" preference, and
- the agents' preferences are always within some distance from the "center" preference.

Thus, we investigate the relation between the possibility of constructing a "nice" social choice function and the degree of similarity of preferences among the agents.

More specifically, we find the range of the degrees of similarity to have a "nondictatorial" social choice function satisfying "unanimity" and "strategy-proofness". The definitions will be give in the next section. Informally, a social choice function is nondictatorial if there is no dictator; is *unanimous* if the complete agreement among the agents is socially respected; is *strategy-proof* if each agent does not have an incentive to misreport his preferences.

2 Basic notation and definitions

2.1 Social choice functions and axioms

Let $N = \{1, \ldots, n\}$ be a finite set of **agents**, and X be a finite set of **alternatives**. Let $m = |X| \geq 3$ be the number of the alternatives. Assume that each agent ranks the alternatives from the best one to the worst one without ties. Such a preference is called a **linear order**.[2] Let L be the set of all linear orders on X. Typical notation for agent i's preference is $R_i \in L$, and for each $x, y \in X$ ($x \neq y$), $x \, R_i \, y$ means that x is preferred to y. For each k ($1 \leq k \leq m$), the kth ranked alternative according to $R_i \in L$ is denoted $r^k(R_i)$.

A list of all agents' preferences $\boldsymbol{R} = (R_1, \ldots, R_n)$ is called a **preference profile**.

[2] Formally, a binary relation Q on X is a linear order if it is *complete* (for each $x, y \in X$ ($x \neq y$), $x \, Q \, y$ or $y \, Q \, x$), *transitive* (for each $x, y, z \in X$, $x \, Q \, y$ and $y \, Q \, z$ imply $x \, Q \, z$), and *antisymmetric* (for each $x, y \in X$, $x \, Q \, y$ and $y \, Q \, x$ imply $x = y$). Usually, completeness is defined in a way which it includes *reflexivity* (for each $x \in X$, $x \, Q \, x$), but in this article whose reader might not be specialists of this area, we drop reflexivity to maintain a simple interpretation of $x \, R_i \, y$: "x is preferred to y".

When agent i changes his preference from R_i to R_i' in $\boldsymbol{R}$, the resulting preference profile is written as $(R_i', \boldsymbol{R}_{-i})$.

For each $D \subset L$, a function f from D^n into X is called a social **choice function on** D, and D is called the **domain**.[3]

A social choice function f on D is

- **unanimous** if for each $x \in X$ and each $\boldsymbol{R} \in D^n$ such that $r^1(R_i) = x$ for each $i \in N$, $f(\boldsymbol{R}) = x$.
- **strategy-proof** if for each $\boldsymbol{R} \in D^n$, each $i \in N$, and each $R_i' \in D$,

$$f(\boldsymbol{R}) = f(R_i', \boldsymbol{R}_{-i}) \text{ or } f(\boldsymbol{R})\ R_i\ f(R_i', \boldsymbol{R}_{-i}).$$

- **dictatorial** if there exists $i^* \in N$ such that for each $\boldsymbol{R} \in D^n$, $f(\boldsymbol{R}) = r^1(R_{i^*})$. This i^* is called the **dictator**.

Unanimity says that if there is the common best alternative, then it should be socially chosen. *Strategy-proofness* says that reporting a false preference (R_i') is never profitable. A social choice function is dictatorial if the social choice is always the best alternative of the dictator. Usually, dictatorship is not an acceptable way of social decision making.

2.2 Kemeny distance

As a domain, we consider the collection of preferences whose distance from the "center preference" is less than or equal to some fixed distance. In the following, let R^* denote the "center preference".

For each $R_i, R_i' \in L$, let $d(R_i, R_i') = |\{\{x, y\}|\ x\ R_i\ y \text{ and } y\ R_i'\ x\}|$; the number of (unordered) pairs of alternatives on which R_i and R_i' disagree. This d is known as **Kemeny distance** (Kemeny, 1959; Kemeny and Snell, 1962). Since there are $\frac{m(m-1)}{2}$

[3] Strictly speaking, the domain is D^n, but in the literature, D is often called the domain.

pairs of alternatives, for each $R, R' \in L$, $0 \leq d(R, R') \leq \frac{m(m-1)}{2}$.

For each k ($0 \leq k \leq \frac{m(m-1)}{2}$), let $D(k) = \{R \in L \mid d(R^*, R) \leq k\}$; the collection of preferences within k distance from the center. This k is a measure of similarity of agents' preferences. The smaller k is, the more similar the agents' preferences are. For example, $D(0) = \{R^*\}$ and $D(\frac{m(m-1)}{2}) = L$. Intuitively, the smaller k is, the easier to construct a desirable social choice function. In the next section, we find when we can construct nondictatorial social choice function satisfying *unanimity* and *strategy-proofness* in terms of k.

3 A result

The starting point of our analysis is the following well known result:

Theorem 1 (Gibbard (1973); Satterthwaite (1975)). *A social choice function on L satisfies* unanimity *and* strategy-proofness *if and only if it is a dictatorship*.

Thus, if $k = \frac{m(m-1)}{2}$, the impossibility appears. If $k = 0$, the center preference R^* is the only preference, and the only *unanimous* social choice function is choosing $r^1(R^*)$. By definition, this is dictatorship, but it is an innocuous dictatorship in the sense that the social choice is the best alternative for each agent.

We want to know what happens when $1 \leq k \leq \frac{m(m-1)}{2} - 1$. The following result is our main theorem. It finds the boundary between dictatorship and nondictatorship in terms of the degrees of similarity of agents' preferences.

Theorem 2. *For each k ($1 \leq k \leq \frac{m(m-1)}{2}$), a nondictatorial social choice function on $D(k)$ satisfying* unanimity *and* strategy-proofness *exists if and only if* $1 \leq k \leq m - 1$.

Proof. Let $r^1(R^*) = x^1$, $r^2(R^*) = x^2, \dots, r^m(R^*) = x^m$. That is, for each h ($1 \leq h \leq m$), x^h is the hth ranked alternative according to the center R^*.

First, assume that $1 \leq k \leq m - 1$. Then, x^{k+1} is top ranked in the only one preference in $D(k)$. Let R' denote this preference:

R': $x^{k+1}\ x^1\ x^2 \dots x^k$ (x^{k+1} is top ranked, x^1 is second ranked, and so on).

Let $i^* \in N$. For each $\boldsymbol{R} \in D(k)^n$, let

$$f(\boldsymbol{R}) = \begin{cases} x^{k+1}, & \text{if } R_{i^*} = R' \text{ and } |\{i \in N \mid x^{k+1}\ R_i\ x^1\}| \geq \frac{1}{2}, \\ x^1, & \text{if } R_{i^*} = R' \text{ and } |\{i \in N \mid x^{k+1}\ R_i\ x^1\} < \frac{1}{2}, \\ r^1(R_{i^*}), & \text{otherwise.} \end{cases}$$

It is easy to see that f satisfies *unanimity*. To show that f is *strategy-proof*, let $\boldsymbol{R} \in D^n$.

If $R_{i^*} \neq R'$, then the social choice is the best alternative according to agent i^*'s preference. Thus, agent i^* has no incentive to misreport. In this case, each agent $i \in N \setminus \{i^*\}$ cannot affect social choice, and hence he has no incentive to misreport.

Thus, it suffices to consider the case $R_{i^*} = R'$. In this case, the social choice is the winner of the simple majority between x^{k+1} and x^1. (The tie is broken by choosing x^{k+1}, but this is not essential.) In this case, it can be seen that each agent $i \in N \setminus \{i^*\}$ has no incentive to misreport. If x^{k+1} is chosen, then it is the best alternative for agent i^*, and hence he does not have an incentive to misreport, either. Thus, assume that x^1 is chosen. At R_{i^*} ($= R'$), x^1 is the second ranked alternative. By reporting a false preference $R'_{i^*} \in D(k) \setminus \{R'\}$, $r^1(R'_{i^*})$ is chosen, but it is never x^{k+1}. (Recall that x^{k+1} is top ranked only by R'.) Since x^{k+1} is the only alternative which is preferred to x^1 according to the true preference R_{i^*}, misreporting R'_{i^*} is not profitable. Therefore, f is *strategy-proof*.

Next, assume that $k = 0$ or $k \geq m$.

When $k = 0$, the only *unanimous* social choice function is always choosing the best alternative according to the center preference. By definition, it is the dictatorship.

Assume $k \geq m$. Then, it can be seen that $D(k)$ is “linked” (Aswal, Chatterji, and Sen, 2003). Aswal, Chatterji, and Sen (2003) show that on each linked domain, each *unanimous* and *startegy-proof* social choice function is dictatorial. ■

As we mention in the proof of Theorem 2, $D(k)$ is “linked” when $k \geq m$. This would be an interesting new example of a “linked” domain. We also note that Theorem 2 is easily shown thanks to the earlier result that on each “linked” domain, each *unanimous* and *strategy-proof* social choice function is dictatorial (Aswal, Chatterji, and Sen, 2003). Without it, it would be hard to prove Theorem 2.

4 Concluding remarks

We completely characterize the degrees of similarity to have a nondictatorial social choice function satisfying *unanimity* and *strategy-proofness*. However, we do not give a complete answer to our motivating question: what is the necessary degree of similarity of preferences to construct a “nice” way of making social choices? Among the nondictatorial social choice functions, some are nearly dictatorship and others treat the agents equally. Actually, the nondictatorial social choice function constructed in the proof of Theorem 2 would be far from ideal in most cases. Thus, finding a complete answer to the above question would be an interesting open question.

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