Maximal Objects and Minimal Objects in the Sets with Operations

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Abstract

Maximal objects and minimal objects in families of subsets are studied by imposing axioms on the families to generalize some common properties of maximal open sets and maximal closed sets, dually those of minimal open sets and minimal closed sets, in topological spaces. Sets with operations κ on families of subsets are studied as examples, and some properties of maximal κ-open sets and minimal κ-closed sets are obtained.

Introduction

In a series of papers [5, 6, 7] we studied some outstanding properties of minimal open sets and maximal open sets and their duals, namely maximal closed sets and minimal closed sets. Recently, various types of maximal open sets and minimal closed sets and their duals are studied by many mathematicians, for example, [15, 14, 1]. Therefore, it seems reasonable to study axiomatic approach to these objects which appear in many fields. The purpose of this paper is to formulate some of the results obtained in [5, 6, 7] imposing an axiom on a family $\mathcal{S} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of a set X.

In Section 1 we consider any family $\mathcal{S} \subset \mathcal{P}(X)$ and define *maximal objects* and *minimal objects* in $\mathcal{S}$. Here, any set $A \in \mathcal{S}$ is called an *object* of $\mathcal{S}$. To generalize some of the results in [5, 6, 7], we consider the following axioms for $\mathcal{S}$ which state that $\mathcal{S}$ is closed under *finite unions* and *finite intersections*, respectively:

Axiom $\mathcal{S}$(FU) If $U, V \in \mathcal{S}$, then $U \cup V \in \mathcal{S}$.

Axiom $\mathcal{S}$(FI) If $U, V \in \mathcal{S}$, then $U \cap V \in \mathcal{S}$.

These axioms are, for example, those of inf (sup) semilattice in Definition O-1.8 of Gierz et al. [2], or join (meet) semilattice in Section 3.1 of Wood [16]. We prove key Lemmas 1.4 and 1.14 with Axioms $\mathcal{S}$(FU) and $\mathcal{S}$(FI), respectively.

Let $\mathcal{F}$ be any family of subsets of a set X such that $\mathcal{F}$ contains at least one non-empty set. An *operation* κ on $\mathcal{F}$ is a function $\kappa : \mathcal{F} \to \mathcal{P}(X)$ such that $U \subset U^\kappa$ for each $U \in \mathcal{F}$, where $U^\kappa = \kappa(U)$ (see Section 2). If $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is an operation, we call a triple $(X, \mathcal{F}, \kappa)$ a *space*. If $(X, \mathcal{F}, \kappa)$ is a space, we call X a $(\mathcal{F}, \kappa)$-space or a κ-space [12]. Let $\kappa : \mathcal{F} \to \mathcal{P}(X)$ be an operation. A subset A of X is called a *κ-open* set of X if for each $x \in A$ there exists a set $U \in \mathcal{F}$ such that $x \in U \subset U^\kappa \subset A$; a subset C of X is called a *κ-closed* set of X if its complement $X - C$ is a κ-open set in X (Definition 2.2).

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In Section 2 we consider the families of κ-open sets and κ-closed sets in a κ-space. Then it is possible to define maximal κ-open sets and minimal κ-closed sets. Since any union of κ-open sets is a κ-open set (see Section 2), the family of κ-open sets and the family of κ-closed sets satisfy Axioms $\mathcal{S}$(FU) and $\mathcal{S}$(FI), respectively. Therefore some results for maximal κ-open sets are obtained by setting $\mathcal{S} = \mathcal{F}_\kappa$ for the family $\mathcal{S}$ in Section 1.1, where $\mathcal{F}_\kappa$ is the family of all κ-open sets; and the dual results for minimal κ-closed sets are obtained by setting $\mathcal{S} = \{\, F \mid X - F \in \mathcal{F}_\kappa \,\}$ for the family $\mathcal{S}$ in Section 1.2. We see that intersections of (finite) κ-open sets are not necessarily κ-open sets, and the results corresponding to Lemmas 1.4 and 1.14 are not proved for maximal κ-closed sets and minimal κ-open sets in general; therefore, these properties of maximal κ-closed sets and minimal κ-open sets are not obtained by the general results in Section 1.

If $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is *regular* (Definition 2.5), then $A_1 \cap A_2 \in \mathcal{F}_\kappa$ for any $A_1, A_2 \in \mathcal{F}_\kappa$ (Proposition 2.6). Therefore, if $\cup\mathcal{F} = X$ and $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is a regular operation, then $\mathcal{F}_\kappa$ is a topology on X, and hence the results in [5, 6, 7] hold for $\mathcal{F}_\kappa$.

1 Maximal objects and minimal objects

In this section we fix a family $\mathcal{S} \subset \mathcal{P}(X)$ to consider maximal objects and minimal objects in $\mathcal{S}$ imposing Axiom $\mathcal{S}$(FU) and Axiom $\mathcal{S}$(FI) on $\mathcal{S}$ in Sections 1.1 and 1.2, respectively. If a subset A of X satisfies $A \in \mathcal{S}$, then A is called an *object* of $\mathcal{S}$. If $A \neq X$, then A is called a *proper* object; if $A \neq \emptyset$, then A is called a *non-empty* object.

1.1 Maximal objects in $\mathcal{S}$.

Definition 1.1. A proper non-empty object $U \in \mathcal{S}$ is called a *maximal object in* $\mathcal{S}$ if any object in $\mathcal{S}$ which contains U is X or U.

We note that in Definition 1.1 we need not assume that $X \in \mathcal{S}$.

Example 1.2. Let $X = \{a,\ b,\ c\}$.
(1) If $\mathcal{S} = \{\emptyset, \{a,b\}, X\}$, then $\{a,b\}$ is a maximal object in $\mathcal{S}$.
(2) If $\mathcal{S} = \{\emptyset, \{a,b\}\}$, then $\{a,b\}$ is a maximal object in $\mathcal{S}$.
(3) If $\mathcal{S} = \{\{a,b\}, X\}$, then $\{a,b\}$ is a maximal object in $\mathcal{S}$.
(4) If $\mathcal{S} = \{\emptyset, X\}$, then there is no maximal object in $\mathcal{S}$.

Example 1.3. Let $[0,2] = \{\, x \mid 0 \le x \le 2 \,\}$, $U = \{\, x \mid 0 < x < 2 \,\}$, $U_n = \{\, x \mid \frac{1}{n} < x < 2 \,\}$ (for any positive integer n) be intervals on the real line. Let $\mathcal{S} = \{\, U_n \mid n \ge 1 \,\} \cup \{\, U \,\}$.
(1) If $\mathcal{S}$ is regarded as $\mathcal{S} \subset \mathcal{P}(X)$ for $X = [0,2]$, then U is a maximal object in $\mathcal{S}$.
(2) If $\mathcal{S}$ is regarded as $\mathcal{S} \subset \mathcal{P}(X)$ for $X = U$, then U is *not* a maximal object in $\mathcal{S}$ and there exists no maximal object in $\mathcal{S}$.

To study some properties of maximal objects in $\mathcal{S}$, we consider the following axiom for $\mathcal{S}$ which states that $\mathcal{S}$ is closed under *finite unions*:

Axiom $\mathcal{S}$(FU) If $U, V \in \mathcal{S}$, then $U \cup V \in \mathcal{S}$.

The following result is obtained by Definition 1.1.

Lemma 1.4. *If $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU)*, then the following results hold.*
(1) *If U is a maximal object in $\mathcal{S}$ and $W \in \mathcal{S}$, then $U \cup W = X$ or $W \subset U$.*
(2) *If U and V are maximal objects in $\mathcal{S}$, then $U \cup V = X$ or $U = V$.*

Example 1.5. Let $U = \{a, b\}$, $V = \{b, c\}$, $W = \{a, c\}$, $X = \{a, b, c\}$ and $Y = \{a, b, c, d\}$.
(1) If we regard $\mathcal{S} = \{U, V, W\} \subset \mathcal{P}(X)$, then U, V, W are maximal objects in $\mathcal{S}$ and $\mathcal{S}$ does not satisfy Axiom $\mathcal{S}$(FU).
(2) If we regard $\mathcal{S} = \{U, V, W, X\} \subset \mathcal{P}(X)$, then U, V, W are maximal objects in $\mathcal{S}$ and $\mathcal{S}$ satisfies Axiom $\mathcal{S}$(FU).
(3) If we regard $\mathcal{S} = \{U, V, W, X\} \subset \mathcal{P}(Y)$, then X is a maximal object in $\mathcal{S}$ and $\mathcal{S}$ satisfies Axiom $\mathcal{S}$(FU).
(4) If we regard $\mathcal{S} = \{U, V\} \subset \mathcal{P}(Y)$, then U, V are maximal objects in $\mathcal{S}$ and $\mathcal{S}$ does not satisfy Axiom $\mathcal{S}$(FU). We see $U \cup V = \{a, b, c\} \neq Y$ and $U \neq V$.

Theorem 1.6. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU). *Let U and U_λ be maximal objects in $\mathcal{S}$ for any element λ of Λ and $U \neq U_\lambda$ for any element λ of Λ. Then $X - \cap_{\lambda\in\Lambda} U_\lambda \subset U$ and hence $\cap_{\lambda\in\Lambda} U_\lambda \neq \emptyset$.*

Proof. Since $X = \cap_{\lambda\in\Lambda}(U \cup U_\lambda) = U \cup (\cap_{\lambda\in\Lambda} U_\lambda)$ by Lemma 1.4(2), we have $X - \cap_{\lambda\in\Lambda} U_\lambda \subset U$. If $\cap_{\lambda\in\Lambda} U_\lambda = \emptyset$, then we have $X = U$, which contradicts our assumption that U is a maximal object in $\mathcal{S}$ and hence $\cap_{\lambda\in\Lambda} U_\lambda \neq \emptyset$. □

Corollary 1.7. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU). *Let U_λ and U_ν be maximal objects in $\mathcal{S}$ for any element λ of Λ and ν of* N. *If there exists an element ν of* N *such that $U_\lambda \neq U_\nu$ for any element λ of Λ, then $\cap_{\lambda\in\Lambda} U_\lambda \not\subset \cap_{\nu\in\mathrm{N}} U_\nu$.*

Proof. Let $\nu' \in \mathrm{N}$ be an element such that $U_\lambda \neq U_{\nu'}$ for any element λ of Λ. If $\cap_{\lambda\in\Lambda} U_\lambda \subset \cap_{\nu\in\mathrm{N}} U_\nu$, then $\cap_{\lambda\in\Lambda} U_\lambda \subset U_{\nu'}$. It follows that $X = (\cap_{\lambda\in\Lambda} U_\lambda) \cup U_{\nu'} \subset U_{\nu'}$ by Theorem 1.6, which contradicts our assumption that $U_{\nu'}$ is a maximal object in $\mathcal{S}$. □

Corollary 1.8. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU). *Let U_λ be a maximal object in $\mathcal{S}$ for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If* N *is a proper non-empty subset of Λ, then*
(1) $(\cap_{\lambda\in\Lambda\setminus\mathrm{N}} U_\lambda) \cup (\cap_{\nu\in\mathrm{N}} U_\nu) = X$, *where $\Lambda \setminus \mathrm{N}$ is the difference of index sets.*
(2) $\cap_{\lambda\in\Lambda} U_\lambda \subsetneqq \cap_{\nu\in\mathrm{N}} U_\nu$.

Proof. (1) is obtained by Theorem 1.6.
(2) If $\overline{\nu} \in \Lambda \setminus \mathrm{N}$, then $U_{\overline{\nu}} \cup (\cap_{\lambda\in\Lambda} U_\lambda) = U_{\overline{\nu}}$ and $U_{\overline{\nu}} \cup (\cap_{\nu\in\mathrm{N}} U_\nu) = X$ by Theorem 1.6. It follows then that if $\cap_{\lambda\in\Lambda} U_\lambda = \cap_{\nu\in\mathrm{N}} U_\nu$, then $X = U_{\overline{\nu}}$, which contradicts our assumption that $U_{\overline{\nu}}$ is a maximal object. □

Corollary 1.9. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU). *Let U_α, U_β, U_γ be maximal objects in $\mathcal{S}$ such that $U_\alpha \neq U_\beta$. If $U_\alpha \cap U_\beta \subset U_\gamma$, then $U_\alpha = U_\gamma$ or $U_\beta = U_\gamma$.*

Proof. Set $\Lambda = \{\alpha, \beta\}$ and $N = \{\gamma\}$ in Corollary 1.7, then the result follows. □

Corollary 1.10. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU). *If U_α, U_β, U_γ are maximal objects in $\mathcal{S}$ which are different from each other, then $U_\alpha \cap U_\beta \not\subset U_\alpha \cap U_\gamma$.*

Proof. Set $\Lambda = \{\alpha, \beta\}$ and $N = \{\alpha, \gamma\}$ in Corollary 1.7, then the result follows. □

We denote by $|\Lambda|$ the cardinality of the index set Λ.

Theorem 1.11 (Decomposition theorem for maximal objects in $\mathcal{S}$)**.** *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU). *Assume that $|\Lambda| \geq 2$ and let U_λ be a maximal object in $\mathcal{S}$ for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then for any element μ of Λ,*

$$U_\mu = (\cap_{\lambda\in\Lambda} U_\lambda) \cup (X - \cap_{\lambda\in\Lambda\setminus\{\mu\}} U_\lambda).$$

Proof. Since $\cap_{\lambda\in\Lambda}U_\lambda = (\cap_{\lambda\in\Lambda\setminus\{\mu\}}U_\lambda)\cap U_\mu$, we have

$$(\cap_{\lambda\in\Lambda}U_\lambda)\cup(X-\cap_{\lambda\in\Lambda\setminus\{\mu\}}U_\lambda) = U_\mu\cup(X-\cap_{\lambda\in\Lambda\setminus\{\mu\}}U_\lambda) = U_\mu$$

by Theorem 1.6. □

Theorem 1.12. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FU). *Assume that $|\Lambda|\geq 2$ and let U_λ be a maximal object in $\mathcal{S}$ for any element λ of Λ and $U_\lambda\neq U_\mu$ for any elements λ and μ of Λ with $\lambda\neq\mu$. If $\cap_{\lambda\in\Lambda}U_\lambda=\emptyset$, then $\{U_\lambda \mid \lambda\in\Lambda\}$ is the set of all maximal objects in $\mathcal{S}$.*

Proof. If there exists another maximal object U_ν in $\mathcal{S}$ which is not equal to U_λ for any element λ of Λ, then $\emptyset = \cap_{\lambda\in\Lambda}U_\lambda = \cap_{\lambda\in(\Lambda\cup\{\nu\})\setminus\{\nu\}}U_\lambda$. However, by Theorem 1.6, we have $\cap_{\lambda\in(\Lambda\cup\{\nu\})\setminus\{\nu\}}U_\lambda\neq\emptyset$, which contradicts our assumption. □

1.2 Minimal objects in $\mathcal{S}$.

Definition 1.13. A proper non-empty object $F\in\mathcal{S}$ is called a *minimal object in $\mathcal{S}$* if any object in $\mathcal{S}$ which is contained in F is $\emptyset$ or F.

We need not assume that $\emptyset\in\mathcal{S}$ in Definition 1.13. We see that $F\in\mathcal{S}$ is a minimal object in $\mathcal{S}$ if and only if $X-F$ is a maximal object in $X-\mathcal{S}=\{X-F \mid F\in\mathcal{S}\}$. Hence the proofs of the statements in this section are obtained by dual arguments of Section 1.1 and they are omitted.

We consider the following axiom on $\mathcal{S}$:

Axiom $\mathcal{S}$(FI) If $U,V\in\mathcal{S}$, then $U\cap V\in\mathcal{S}$.

Lemma 1.14. *If $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI)*, then the following results hold.*
(1) *If F is a minimal object in $\mathcal{S}$ and $N\in\mathcal{S}$, then $F\cap N=\emptyset$ or $F\subset N$.*
(2) *If F and S are minimal objects in $\mathcal{S}$, then $F\cap S=\emptyset$ or $F=S$.*

Example 1.15. Let $U=\{a,b\}$, $V=\{b,c\}$ and $X=\{a,b,c\}$.
(1) If we regard $\mathcal{S}=\{U,V\}\subset\mathcal{P}(X)$, then U, V are minimal objects in $\mathcal{S}$ and $\mathcal{S}$ does not satisfy Axiom $\mathcal{S}$(FI).
(2) If we regard $\mathcal{S}=\{\{b\},U,V\}\subset\mathcal{P}(X)$, then $\{b\}$ is a minimal object in $\mathcal{S}$ and $\mathcal{S}$ satisfies Axiom $\mathcal{S}$(FI).
(3) If we regard $\mathcal{S}=\{\emptyset,\{b\},U,V\}\subset\mathcal{P}(X)$, then $\{b\}$ is a minimal object in $\mathcal{S}$ and $\mathcal{S}$ satisfies Axiom $\mathcal{S}$(FI) and $\emptyset\in\mathcal{S}$.

Theorem 1.16. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI)*. Let F and F_λ be minimal objects in $\mathcal{S}$ for any element λ of Λ and $F\neq F_\lambda$ for any element λ of Λ. Then $F\subset X-\cup_{\lambda\in\Lambda}F_\lambda$ and hence $\cup_{\lambda\in\Lambda}F_\lambda\neq X$.*

Corollary 1.17. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI)*. Let F_λ and F_ν be minimal objects in $\mathcal{S}$ for any element λ of Λ and ν of N. If there exists an element ν of N such that $F_\lambda\neq F_\nu$ for any element λ of Λ, then $\cup_{\nu\in\mathrm{N}}F_\nu\not\subset\cup_{\lambda\in\Lambda}F_\lambda$.*

Corollary 1.18. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI)*. Let F_λ be a minimal object in $\mathcal{S}$ for any element λ of Λ and $F_\lambda\neq F_\mu$ for any elements λ and μ of Λ with $\lambda\neq\mu$. If N is a proper non-empty subset of Λ, then*
(1) $(\cup_{\lambda\in\Lambda\setminus\mathrm{N}}F_\lambda)\cap(\cup_{\nu\in\mathrm{N}}F_\nu)=\emptyset$.
(2) $\cup_{\nu\in\mathrm{N}}F_\nu\subsetneqq\cup_{\lambda\in\Lambda}F_\lambda$.

Corollary 1.19. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI). *Let F_α, F_β, F_γ be minimal objects in $\mathcal{S}$ such that $F_\alpha \neq F_\beta$. If $F_\alpha \cup F_\beta \supset F_\gamma$, then $F_\alpha = F_\gamma$ or $F_\beta = F_\gamma$.*

Corollary 1.20. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI). *If F_α, F_β, F_γ are minimal objects in $\mathcal{S}$ which are different from each other, then $F_\alpha \cup F_\beta \not\supset F_\alpha \cup F_\gamma$.*

Theorem 1.21 (Recognition principle for minimal objects in $\mathcal{S}$). *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI). *Assume that $|\Lambda| \geq 2$ and let F_λ be a minimal object in $\mathcal{S}$ for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then for any element μ of Λ,*

$$F_\mu = (\cup_{\lambda\in\Lambda} F_\lambda) \cap (X - \cup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda).$$

Theorem 1.22. *Assume that $\mathcal{S}$ satisfies* Axiom $\mathcal{S}$(FI). *Assume that $|\Lambda| \geq 2$ and let F_λ be a minimal object in $\mathcal{S}$ for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\cup_{\lambda\in\Lambda} F_\lambda = X$, then $\{F_\lambda | \lambda \in \Lambda\}$ is the set of all minimal objects in $\mathcal{S}$ of X.*

2 Maximal κ-open sets and minimal κ-closed sets

Let $\mathcal{P}(X)$ be the power set of a set X and $\mathcal{F} \subset \mathcal{P}(X)$ such that $\mathcal{F}$ contains at least one non-empty set. However, we do *not* assume that $\cup\mathcal{F} := \cup_{U\in\mathcal{F}} U = X$. An *operation* κ on $\mathcal{F}$ is a function

$$\kappa : \mathcal{F} \to \mathcal{P}(X)$$

such that $U \subset U^\kappa$ for each $U \in \mathcal{F}$, where $U^\kappa = \kappa(U)$. We call a triple $(X, \mathcal{F}, \kappa)$ a *space* for any operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$; if $(X, \mathcal{F}, \kappa)$ is a space, we call X a $(\mathcal{F}, \kappa)$-*space* or a κ-*space* [12].

Remark 2.1. Let τ be any family of sets. Kasahara [3] defined an operation α as a function $\alpha : \tau \to \mathcal{P}(\cup\tau)$ such that $G \subset G^\alpha$ for any $G \in \tau$, where $\cup\tau$ is the union of the sets in τ. We remove the restriction $\cup\tau$ in $\mathcal{P}(\cup\tau)$ to define our operation in [12] and in this paper.

Definition 2.2. ([12]) Let $\kappa : \mathcal{F} \to \mathcal{P}(X)$ be an operation. A subset A of X is called a κ-*open* set of X if for each $x \in A$ there exists a set $U \in \mathcal{F}$ such that $x \in U \subset U^\kappa \subset A$. The family of all κ-open sets is denoted by $\mathcal{F}_\kappa$. A subset C of X is called a κ-*closed* set of X if its complement $X - C$ is a κ-open set in X.

If $A_\lambda \in \mathcal{F}_\kappa$ for any λ of Λ, then $\cup_{\lambda\in\Lambda} A_\lambda \in \mathcal{F}_\kappa$ by Proposition 2.8 of [12]. If we set $\mathcal{S} = \mathcal{F}_\kappa$ in Definition 1.1 for any space $(X, \mathcal{F}, \kappa)$, then a maximal object in $\mathcal{S}$ is a *maximal κ-open set*; moreover, since $\mathcal{S} = \mathcal{F}_\kappa$ satisfies Axiom $\mathcal{S}$(FU), we have the results for maximal κ-open sets by theorems and corollaries for maximal objects in Section 1.1. If $\mathcal{S} = \{ F \mid X - F \in \mathcal{F}_\kappa \}$ in Definition 1.13 for a space $(X, \mathcal{F}, \kappa)$, then a minimal object in $\mathcal{S}$ is a *minimal κ-closed set.* The family $\mathcal{S} = \{ F \mid X - F \in \mathcal{F}_\kappa \}$ satisfies Axiom $\mathcal{S}$(FI), since $\mathcal{F}_\kappa$ satisfies Axiom $\mathcal{S}$(FU), and the results for minimal κ-closed sets are obtained by the results in Section 1.2.

Example 2.3. Let U be a subset of a set X such that $\emptyset \subsetneqq U \subsetneqq X$ and $\mathcal{F} = \{U\}$.
(1) If $\kappa(U) = X$, then $\mathcal{F}_\kappa = \{\emptyset\}$.
(2) If $\kappa(U) = U$, then $\mathcal{F}_\kappa = \{\emptyset, U\}$.

We consider the following axiom for the set $\mathcal{F}$, which states that $\mathcal{F}$ is closed under *finite intersections* and that $\mathcal{F}$ is closed under *arbitrary unions*, respectively:

Axiom $\mathcal{F}$(FI) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Axiom $\mathcal{F}$(AU) If $A_\lambda \in \mathcal{F}$ *for any* $\lambda \in \Lambda$, *then* $\cup_{\lambda\in\Lambda} A_\lambda \in \mathcal{F}$.

Then the following results are immediate consequences of the definition of the κ-open set.

Proposition 2.4. *Let* $\kappa : \mathcal{F} \to \mathcal{P}(X)$ *be an operation. If* $\mathcal{F}$ *satisfies Axiom* $\mathcal{F}$(AU)*, then the following results hold.*
(1) $\mathcal{F}_\kappa \subset \mathcal{F} \cup \{\emptyset\}$.
(2) *If* $U^\kappa = U$ *for any* $U \in \mathcal{F}$*, then* $\mathcal{F}_\kappa = \mathcal{F} \cup \{\emptyset\}$.

Definition 2.5. An operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is *regular* if for any $a \in X$ and any sets $U, V \in \mathcal{F}$ with $a \in U \cap V$, there exists a set $W \in \mathcal{F}$ such that $a \in W \subset W^\kappa \subset U^\kappa \cap V^\kappa$. An operation $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is *monotone* if $U \subset V$ and $U, V \in \mathcal{F}$ implies $U^\kappa \subset V^\kappa$ (cf. p.98 of [3]).

We have the following results by the arguments similar to the proofs of Proposition 2.9(1) of [13] and p.98 of [3].

Proposition 2.6. (1) *If* $\kappa : \mathcal{F} \to \mathcal{P}(X)$ *is regular, then* $A_1 \cap A_2 \in \mathcal{F}_\kappa$ *for any* $A_1, A_2 \in \mathcal{F}_\kappa$.
(2) *If an operation* $\kappa : \mathcal{F} \to \mathcal{P}(X)$ *is monotone and* $\mathcal{F}$ *satisfies* Axiom $\mathcal{F}$(FI)*, then* κ *is regular.*

If $\mathcal{F}$ satisfies Axiom $\mathcal{F}$(AU) and $\emptyset \in \mathcal{F}$, then we have $\mathcal{F}_\kappa = \mathcal{F}$ by Proposition 2.4(2) for the κ defined there. If $\cup\mathcal{F} = X$ and $\kappa : \mathcal{F} \to \mathcal{P}(X)$ is a regular operation, then $\mathcal{F}_\kappa$ is a topology on X by Proposition 2.6(1), and hence the results in [5, 6, 7] hold for minimal and maximal κ-open sets and maximal and minimal κ-closed sets.

Remark 2.7. When $\gamma : \tau \to \mathcal{P}(X)$ is the operation in Kasahara [3] for some topological space (X, τ), we studied maximal γ-open sets and its dual, minimal γ-closed sets in [4, 8, 9, 11] to generalize the results in [5, 6, 7]. Some of the results in these articles are generalized in this paper making use of the operation κ which is more general than those studied in previous articles. More precisely:

If we set $\mathcal{S} = \tau_\gamma$ for an operation $\gamma : \tau \to \mathcal{P}(X)$ for some topological space (X, τ), then Lemma 1.4 implies Lemma 2.2 of [8]; Theorems 1.6, 1.11 and 1.12 imply Theorems 2.4, 2.7 and 2.8 of [11], respectively; Corollaries 1.7 and 1.8 imply Corollaries 2.5 and 2.6 of [11]; Corollaries 1.9 and 1.10 imply Theorems 2.3 and 2.4 of [8].

If $\gamma : \tau \to \mathcal{P}(X)$ is an operation for some topological space (X, τ) and the family $\mathcal{S}$ is defined by $\mathcal{S} = \{\, F \mid X - F \in \tau_\gamma \,\}$, then Lemma 1.14 implies Lemma 2.2 of [9]; Theorems 1.16, 1.21 and 1.22 imply Theorems 2.3, 2.8 and 2.9 of [9]; Corollaries 1.17 and 1.18 imply Corollaries 2.5, 2.4 and Theorem 2.6 of [9].

In the attempts to generalize these results further, we studied some aspects of maximal objects and minimal objects in topological spaces in [4, 10].

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