Generators of Invariant Rings and Modular Representations of Symmetric Groups

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Abstract

The Weyl group $W(G)$ of a compact connected Lie group G is a reflection group over $\mathbb{Z}$. Considering the modular case of this integral representation, we will ask if each ring of invariants for the dual representation of $W(SU(n))$ is a polynomial algebra. Our method concerns some generators of the invariant ring. As a corollary, it gives another proof for the result that both $H^*(BT^4;\mathbb{F}_5)^{W(SU(5))^*}$ and $H^*(BT^5;\mathbb{F}_2)^{W(SU(6))^*}$ are not polynomial algebras.

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The Weyl group $W(G)$ of a compact connected Lie group G acts on a maximal torus T^n, and the integral representation $W(G) \longrightarrow GL(n,\mathbb{Z})$ makes $W(G)$ a reflection group, [5] and [8]. If $W(G)^*$ denote the dual representation of $W(G)$, we see $W(PU(n)) = W(SU(n))^*$, [3]. A result of Dwyer–Wilkerson [1] implies that, if $p \geq 5$, the invariant ring $H^*(BT^{p-1};\mathbb{F}_p)^{W(SU(p))^*}$ is not a polynomial algebra. For $p=5$, we will give another proof for the result. Our method concerns some generators of the invariant ring. We hope these generators can be used to find the structure of this ring for at least lower degree elements.

Theorem 1 *The following hold:*

(1) *The invariant ring $H^*(BT^4;\mathbb{F}_5)^{W(SU(5))^*}$ has no generator of algebraic degree n, if $n = 2, 3, 5$.*

(2) *This invariant ring is not a polynomial ring.*

Suppose that V is a finite dimensional vector space over the finite field $\mathbb{F}_p$ and that W is a subgroup of $\mathrm{Aut}(V)$. Let U be a subset of V, and W_U the subgroup of W consisting of elements which fix U pointwise. If dim $V = n$, then the symmetric algebra $S(V)$ is isomorphic to $H^*(BT^n;\mathbb{F}_p) = \mathbb{F}_p[t_1, t_2, \cdots, t_n]$. Although the topological degree of each t_i is 2, as usual, its algebraic degree is

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assumed to be 1. For example, we have $\deg(t_1) = 1, \deg(t_1 t_2) = 2$ and so on. According to [8, Proposition 4.4.3], if $H^*(BT^n; \mathbb{F}_p)^W = \mathbb{F}_p[x_1, x_2, \cdots x_n]$, then $|W| = \prod_{i=1}^{n} \deg(x_i)$. Other necessary conditions for the invariant ring being a polynomial algebra can be found in [7, §5.7].

Theorem 2 *Suppose $V = \bigoplus^5 \mathbb{F}_2$ and $W = W(SU(6))$. Take U to be the 1-dimensional subspace spanned by the vector $\boldsymbol{x} = {}^t(1, 0, 1, 0, 1)$. Then the following hold:*

(1) *The invariant ring $S(V)^{W_U}$ has a generator of algebraic degree n,*

if $n = 1, 2, 3, 5$.

(2) *This invariant ring is not a polynomial ring.*

By [1, Theorem 1.4], if $S(V)^{W^*}$ is a polynomial ring over $\mathbb{F}_p$, so is $S(V)^{W_U}$. So the following is an easy consequence.

Corollary 3 *The invariant ring $H^*(BT^5; \mathbb{F}_2)^{W(SU(6))^*}$ is not a polynomial ring.*

The above result was first proved in [6] by the second author using reflection groups. Here we ask if $H^*(BT^{n-1}; \mathbb{F}_p)^{W(SU(n))^*}$ is a polynomial algebra for $p = 2, 3$. By [6] and [3], we see that the invariant ring is a polynomial algebra if $(p, n) = (2, 2), (2, 4), (3, 3)$, and that it is not a polynomial algebra if $(p, n) = (2, 6), (2, 8), (3, 6), (3, 9)$.

1 The invariant ring $H^*(BT^4; \mathbb{F}_5)^{W(SU(5))^*}$

We will prove Theorem 1 in this section. To do so, we need a few basic results, [2], [4] and [7]. Recall that the representation of $\Sigma_n = W(SU(n))$ is generated by the permutation matrices Σ_{n-1} together with the following $(n-1) \times (n-1)$ matrix:

$$\tau = \begin{pmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & & -1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

In other words, each column vector is one of the set of the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{n-1}\}$ and the vector $\boldsymbol{b} = {}^t(-1, -1, \cdots, -1)$. The transpositions $(i, i+1)$ for $1 \leq i \leq n-1$ generate the symmetric group Σ_n.

If $W = W(SU(5))^*$ and $V = \bigoplus_{i=1}^{4} \mathbb{F}_5\langle t_i\rangle$, the vector $c_1 = t_1 + t_2 + t_3 + t_4$ is $W-$ invariant. With respect to the basis $\{c_1, t_1, t_2, t_3\}$, the corresponding matrix presentations are as follows:

$$(1,2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ (2,3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$(3,4) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \tau = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Lemma 1.1 *Let $V_0 = V/c_1$ and $W = W(SU(5))^*$. If $\pi : S(V) \longrightarrow S(V_0)$ is the map induced by the projection $V \longrightarrow V_0$, then $\pi(S(V)^W) \subset S(V_0)^W$.*

Proof If $x \in S(V)^W$, we write $x = x_0 + c_1x_1$ where x_0 is not divisible by c_1. Then we see $x = wx = wx_0 + c_1wx_1$ for any $w \in W$. Thus $x_0 = wx_0$ modulo (c_1). Consequently, $\pi(x) \in S(V_0)^W$. □

We notice here that $S(V_0)^W = S(V_0)^{\langle W(SU(4)),R\rangle}$, where

$$R = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Let $S(V_0)^{\Sigma_3} = \mathbb{F}_5[\overline{c_1}, \overline{c_2}, \overline{c_3}]$ with $\overline{c_1} = t_1 + t_2 + t_3$, $\overline{c_2} = t_1t_2 + t_1t_3 + t_2t_3$ and $\overline{c_3} = t_1t_2t_3$.

Lemma 1.2 *We have $S(V_0)^{W(SU(4))} = \mathbb{F}_5[u_2, u_3, u_4]$, where $u_2 = \overline{c_2} - \overline{c_1}^2$, $u_3 = \overline{c_3} - \overline{c_1}\,\overline{c_2}$, and $u_4 = \overline{c_1}\,\overline{c_3}$.*

Proof Since $S(V_0)^{W(SU(4))} = S(V)^{\Sigma_4}/c_1$, we obtain the desired results. For example, we get:

$$\begin{aligned}u_2 = \pi(c_2) =& t_1t_2 + t_1t_3 + t_1(-t_1 - t_2 - t_3) + t_2t_3 + t_2(-t_1 - t_2 - t_3) \\ &+ t_3(-t_1 - t_2 - t_3) \\ =& - t_1^2 - t_2^2 - t_3^2 - t_1t_2 - t_1t_3 - t_2t_3 \\ =&(t_1t_2 + t_1t_3 + t_2t_3) - (t_1 + t_2 + t_3)^2 \\ =&\overline{c_2} - \overline{c_1}^2 \ .\end{aligned}$$

□

Lemma 1.3 *We have the following:*

(1) $R(\overline{c_1}) = -\overline{c_1}$, $R(\overline{c_2}) = \overline{c_2}$, *and* $R(\overline{c_3}) = \overline{c_3} + \overline{c_1}\,\overline{c_2} + 2\overline{c_1}^3$.

(2) $R(u_2) = u_2$, $R(u_3) = u_3 + 2\overline{c_1}^3 - 2\overline{c_1}\,\overline{c_2}$, *and* $R(u_4) = -u_4 - \overline{c_1}^2\overline{c_2} - 2\overline{c_1}^4$.

Proof Note that $R(t_i) = t_i + \overline{c_1}$ for $1 \le i \le 3$. Consider the following polynomial:

$$\begin{aligned}\prod_{i=1}^{3}(1 + (t_i + \overline{c_1})X) =& \sum_{i=0}^{3} \overline{c_i}X^i(1 + \overline{c_1}X)^{3-i} \\ =& \sum_{i=0}^{3} \overline{c_i}X^i \sum_{j=0}^{3-i} \binom{3-i}{j}(\overline{c_1}X)^j \\ =& \sum_{k=0}^{3} \sum_{i+j=k} \binom{3-i}{j}\overline{c_i}\,\overline{c_1}^j X^k\end{aligned}$$

Thus $R(\overline{c_2})$, for example, is as follows:

$$\begin{aligned}\sum_{i+j=2} \binom{3-i}{j}\overline{c_i}\,\overline{c_1}^j =& \binom{3-2}{0}\overline{c_2} + \binom{3-1}{1}\overline{c_1}^2 + \binom{3}{2}\overline{c_1}^2 \\ =& \overline{c_2} \mod 5\end{aligned}$$

The other cases are similarly obtained. □

Lemma 1.4 *For* $c_2 = \sum_{1 \le i < j \le 4} t_it_j \in S(V)$, *we have* $\tau(c_2) = c_2 + c_1t_4$.

Proof Let $S\left(\bigoplus_{i=1}^{3}\mathbb{F}_5\langle t_i\rangle\right)^{\Sigma_3} = \mathbb{F}_5[\overline{c_1},\overline{c_2},\overline{c_3}]$ with $\overline{c_1} = t_1+t_2+t_3$, $\overline{c_2} = t_1t_2+t_1t_3+t_2t_3$ and $\overline{c_3} = t_1t_2t_3$. Then it follows:

$$\begin{aligned}\prod_{i=1}^{3}(1+(t_i-t_4)X)(1-t_4X) =&(1-t_4X)\sum_{i=0}^{3}\overline{c_i}X^i(1-t_4X)^{3-i}\\ =&\sum_{i=0}^{3}\overline{c_i}X^i\sum_{j=0}^{4-i}\binom{4-i}{j}(-t_4X)^j\end{aligned}$$

Thus we obtain:

$$\begin{aligned}\tau(c_2) =& \binom{4-2}{0}\overline{c_2}+\binom{3}{1}\overline{c_1}(-t_4)+\binom{4}{2}t_4^2\\ =&\sum_{1\le i<j\le 3}t_it_j+2\left(\sum_{i=1}^{3}t_i\right)t_4+t_4^2\\ =&c_2+c_1t_4\end{aligned}$$

□

Proof of Theorem 1 (1) Lemma 1.3 (2) implies that the invariant ring $S(V_0)^W$ has no generator of degree $n = 3, 5$. Hence, by Lemma 1.1, we see that $H^*(BT^4;\mathbb{F}_5)^{W(SU(5))^*}$ has no generator of degree $n = 3, 5$. We also see that, by Lemma 1.4, this ring has no generator of degree 2.

(2) Suppose the ring were a polynomial ring so that $H^*(BT^4;\mathbb{F}_5)^{W(SU(5))^*} = \mathbb{F}_5[x_1,x_2,x_3,x_4]$. Since, there are no generator of degree n, if $n = 2, 3, 5$, the product $\prod_{i=1}^{4}\deg(x_i)$ has to be at least $1\cdot 4\cdot 6\cdot 10$, which is larger than $|W(SU(5))| = 5!$. This contradiction completes the proof. □

We note here that $u_3^2 - 2u_2u_4 \in S(V_0)^W$. This means that there may be a generator of degree 6 in $H^*(BT^4;\mathbb{F}_5)^{W(SU(5))^*}$.

2 The invariant ring $H^*(BT^5;\mathbb{F}_2)^{W_U}$

We will prove Theorem 2 in this section. Recall that U is the 1-dimensional subspace spanned by the vector $\boldsymbol{x} = {}^t(1,0,1,0,1)$. If

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

then we see $A\boldsymbol{x} = \boldsymbol{x}$ and $A^6 = I$, and $A \in W_U$. Clearly the transpositions $(1,3)$, $(1,5)$ and $(2,4)$ are contained also in W_U. Let H denote the subgroup of W_U generated by these transpositions. For $H = \Big\langle (1,3),(1,5),(2,4) \Big\rangle$ and $K = \mathbb{Z}/6\langle A\rangle$, we notice that H and K are subgroups of W_U, and $H \cap K$ is equal to $\{e\}$ so that $|H \cdot K| = 72$. In fact, we can see $|W_U| = 72$ and $W_U = H \cdot K$. Let $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4, \boldsymbol{b}_5) \in W_U$, where $\boldsymbol{b}_i$ is the i–th column of the matrix B. If, for example, $\boldsymbol{b}_1 = \boldsymbol{b} = {}^t(1,1,1,1,1)$, then we see that $\{\boldsymbol{b}_3, \boldsymbol{b}_5\} = \{\boldsymbol{e}_2, \boldsymbol{e}_4\}$ and $\{\boldsymbol{b}_2, \boldsymbol{b}_4\} \subset \{\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_5\}$. And, if $\boldsymbol{b}_1 = \boldsymbol{e}_1$, then it follows that $\{\boldsymbol{b}_3, \boldsymbol{b}_5\} = \{\boldsymbol{e}_3, \boldsymbol{e}_5\}$ and $\{\boldsymbol{b}_2, \boldsymbol{b}_4\} \subset \{\boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{b}\}$. Hence, we conclude $|W_U| = (3! \times 3!) \times 2 = 72$.

The ring of invariants $S(V)^{W_U} = H^*(BT^5;\mathbb{F}_2)^{W_U}$ is graded. So, we let $S(V)^{W_U} = \bigoplus_{n=0}^{\infty} R_n$. The elements of R_n are said to be homogeneous elements of degree n.

Lemma 2.1 *A basis of the vector space R_1 of degree 1 elements is given by x_1, where $x_1 = t_1 + t_3 + t_5$.*

Proof Since x_1 is invariant under the action of H, any element $x \in R_1$ can be expressed as $x = at_1 + bt_2 + at_3 + bt_4 + at_5$ for suitable $a, b \in \mathbb{F}_2$. Furthermore, x_1 is A–invariant. We see that

$$\begin{aligned} A(x) &= at_2 + bt_3 + at_4 + bt_5 + a(t_1 + t_2 + t_3 + t_4 + t_5) \\ &= at_1 + 2at_2 + (a+b)t_3 + 2at_4 + (a+b)t_5 \\ &= at_1 + (a+b)t_3 + (a+b)t_5 \mod 2. \end{aligned}$$

Hence, if x is W_U–invariant, then $a + b = a$. It follows that $b = 0$. This completes the proof. □

Lemma 2.2 *Let $x_2 = t_2^2 + t_4^2 + t_1t_2 + t_1t_3 + t_1t_4 + t_1t_5 + t_2t_3 + t_2t_4 + t_2t_5 + t_3t_4 + t_3t_5 + t_4t_5$. Then a basis of R_2 is given by $\{x_1^2, x_2\}$. In other words $R_2 = \mathbb{F}_2\langle x_1^2\rangle \oplus \mathbb{F}_2\langle x_2\rangle$.*

Proof We notice that x_2 is H–invariant. Furthermore x_2 is A–invariant:

$$\begin{aligned} A(x_2) &= t_3^2 + t_5^2 + t_2t_3 + t_2t_4 + t_2t_5 + t_2(t_1 + t_2 + t_3 + t_4 + t_5) \\ &\quad + t_3t_4 + t_3t_5 + t_3(t_1 + t_2 + t_3 + t_4 + t_5) \\ &\quad + t_4t_5 + t_4(t_1 + t_2 + t_3 + t_4 + t_5) + t_5(t_1 + t_2 + t_3 + t_4 + t_5) \\ &= t_2^2 + t_4^2 + t_1t_2 + t_1t_3 + t_1t_4 + t_1t_5 + t_2t_3 + t_2t_4 + t_2t_5 \\ &\quad + t_3t_4 + t_3t_5 + t_4t_5 \\ &= x_2 \end{aligned}$$

Hence x_2 is an element of $S(V)^{W_U}$.

Next, we claim that x_2 can not be decomposed in the W_U–invariant. This means that x_2 can not be equal to $y_2 + z_2$ for any non-zero elements y_2 and z_2 in $S(V)^{W_U}$. We notice that x_2 can be divided as a sum of the three elements α_1, α_2 and α_3, where $\alpha_1 = t_1t_3 + t_1t_5 + t_3t_5$, $\alpha_2 = t_2^2 + t_4^2 + t_2t_4$ and $\alpha_3 = t_1t_2 + t_1t_4 + t_2t_3 + t_2t_5 + t_3t_4 + t_4t_5$. However, applying A, we see that each $\alpha_i + \alpha_j$ is not W_U–invariant for $(i,j) = (1,2)$, $(1,3)$ and $(2,3)$. This completes the proof □

Lemma 2.3 *The set* $\{x_1^3, x_1x_2, x_3\}$ *is a basis of* R_3, *where* $x_3 = t_1^3 + t_2^3 + t_3^3 + t_4^3 + t_5^3 + (t_1 + t_2 + t_3 + t_4 + t_5)^3$.

Proof An argument similar to the one used in the proof of Lemma 2.2 shows that x_3 is W_U–invariant, and that x_3 can not be expressed as $y_3 + z_3$ for any non-zero elements y_3 and z_3 in $S(V)^{W_U}$. Consequently, it remains to show that x_3 is not contained in the algebra generated by x_1 and x_2. Here, assume that $x_3 = ax_1^3 + bx_1x_2$. Replacing t_1, t_3 and t_5 by 0, we see $x_1 = 0$. Thus, the right hand side is zero. Considering the left hand side, we see that:

$$\begin{aligned} x_3 &= t_2^3 + t_4^3 + (t_2 + t_4)^3 \\ &= t_2^3t_4^3 + (t_2 + t_4)(t_2 + t_4)^2 \\ &= t_2^3 + t_4^3 + (t_2 + t_4)(t_2^2 + t_4^2) \\ &= t_2^3 + t_4^3 + t_2^3 + t_2^2t_4 + t_2t_4^2 + t_4^3 \\ &= t_2^3 + t_2^2t_4 + t_2t_4^2 \neq 0 \ . \end{aligned}$$

Thus, $x_3 \notin \langle x_1, x_2 \rangle$. Therefore, this contradiction completes the proof. □

Lemma 2.4 *Let* $x_5 = t_1^5 + t_2^5 + t_3^5 + t_4^5 + t_5^5 + (t_1 + t_2 + t_3 + t_4 + t_5)^5$. *This element of* R_5 *is not contained in the algebra generated by* $\{x_1, x_2, x_3, x_4\}$ *for any* $x_4 \in R_4$.

Proof The proof is again similar to the previous one. Assume that $x_5 = ax_1^5+bx_1^3x_2+cx_1^2x_3+dx_1^2x_2^2+ex_1x_4+fx_2x_3$ for $a,b,c,d,e,f \in \mathbb{F}_2$. If $t_5 = t_1+t_3$, then $x_1 = 0$. We calculate the left hand side:

$$\begin{aligned} x_5 &= t_1^5 + t_2^5 + t_3^5 + t_4^5 + (t_1+t_3)^5 + (t_1+t_2+t_3+t_4+t_1+t_3)^5 \\ &= t_1^5 + t_2^5 + t_3^5 + t_4^5 + (t_1+t_3)^5 + (t_2+t_4)^5 \\ &= t_1^5 + t_2^5 + t_3^5 + t_4^5 + t_1^5 + t_1^4t_3 + t_1t_3^4 + t_3^5 + t_2^5 + t_2^4t_4 + t_2t_4^4 + t_4^5 \\ &= t_1^4t_3 + t_1t_3^4 + t_2^4t_4 + t_2t_4^4 \ . \end{aligned}$$

Also the right hand side is fx_2x_3 since $x_1 = 0$. Now, we calculate x_2 and x_3:

$$\begin{aligned} x_2 &= t_2^2 + t_4^2 + t_1t_2 + t_1t_3 + t_1t_4 + t_1t_5 + t_2t_3 + t_2t_4 + t_2t_5 + t_3t_4 + t_3t_5 + t_4t_5 \\ &= t_2^2 + t_4^2 + t_1t_2 + t_1t_3 + t_1t_4 + t_1(t_1+t_3) + t_2t_3 + t_2t_4 + t_2(t_1+t_3) \\ &\quad + t_3t_4 + t_3(t_1+t_3) + t_4(t_1+t_3) \\ &= t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_1t_3 + t_2t_4 \end{aligned}$$

and

$$\begin{aligned} x_3 &= t_1^3 + t_2^3 + t_3^3 + t_4^3 + (t_1+t_3)^3 + (t_1+t_2+t_3+t_4+t_1+t_3)^3 \\ &= t_1^3 + t_2^3 + t_3^3 + t_4^3 + (t_1+t_3)^3 + (t_2+t_4)^3 \\ &= t_1^3 + t_2^3 + t_3^3 + t_4^3 + t_1^3 + t_1^2t_3 + t_1t_3^2 + t_3^3 + t_2^3 + t_2^2t_4 + t_2t_4^2 + t_4^3 \\ &= t_1^2t_3 + t_1t_3^2 + t_2^2t_4 + t_2t_4^2. \end{aligned}$$

It follows that

$$\begin{aligned} x_2x_3 &= (t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_1t_3 + t_2t_4)(t_1^2t_3 + t_1t_3^2 + t_2^2t_4 + t_2t_4^2) \\ &= t_1^4t_3 + t_1^3t_3^2 + t_1^2t_2^2t_4 + t_1^2t_2t_4^2 + t_1^2t_2^2t_3 + t_1t_2^2t_3 + t_2^4t_4 + t_2^3t_4^2 \\ &\quad + t_1^2t_3^3 + t_1t_3^4 + t_2^2t_3^2t_4 + t_2t_3^2t_4^2 + t_1^2t_3t_4^2 + t_1t_3^2t_4^2 + t_2^2t_4^3 + t_2t_4^4 \\ &\quad + t_1^3t_3^2 + t_1^2t_3^3 + t_1t_2^2t_3t_4 + t_1t_2t_3t_4^2 + t_1^2t_2t_3t_4 + t_1t_2t_3^2t_4 + t_2^3t_4^2 + t_2^2t_4^3 \\ &= t_1^4t_3 + t_1t_3^4 + t_2^4t_4 + t_2t_4^4 \\ &\quad + t_1^2t_2^2t_4 + t_1^2t_2t_4^2 + t_1^2t_2^2t_3 + t_1t_2^2t_3^2 + t_2^2t_3^2t_4 + t_2t_3^2t_4^2 + t_1^2t_3t_4^2 + t_1^2t_3^2t_4^2 \\ &\quad + t_1^2t_2t_3t_4 + t_1t_2^2t_3t_4 + t_1t_2t_3^2t_4 + t_1t_2t_3t_4^2. \end{aligned}$$

Thus, $x_5 \neq fx_2x_3$ modulo (x_1). This contradiction completes the proof. □

Proof of Theorem 2 (1) The desired result follows from Lemma 2.1, Lemma 2.2, Lemma 2.3, and Lemma 2.4.

(2) Suppose $S(V)^{W_U}$ were a polynomial ring. Since Lemma 2.4 tells us that $S(V)^{W_U}$ has a generator of degree 5, if $H^*(BT^5;\mathbb{F}_2)^{W_U} = \mathbb{F}_2[x_1,x_2,x_3,x_5,y]$, we would see $|W_U| = 30 \times \deg(y)$. However, we recall that $|W_U| = 72$. This contradiction shows that $S(V)^{W_U}$ is not a polynomial ring. □

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