A Discrete Analogue of the Affine Bäcklund Transformation

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Abstract

The purpose of this paper is to give a discrete analogue of affine minimal surfaces that would be recovered by Lelieuvre's formula from conormal maps, and to consider their affine Bäcklund transformations.

1 Introduction

In these few years, substantial progress has been made in understanding nonlinear partial difference equations from discrete geometric viewpoints. For example, it is known that Hirota's discrete sine-Gordon equation arises as the compatibility condition of discrete constant negative Gaussian curvature surfaces ([1], [10]). In the context of affine differential geometry, discrete proper affine spheres give a geometric interpretation to the discrete Tzitzéica equation ([3], [2]), and discrete improper affine spheres to Hirota's discrete Liouville equation ([8]). Affine minimal surfaces are those for which the affine mean curvature vanishes, and hence include every improper affine sphere. In this paper, we discretize affine minimal surfaces as a continuation of our paper [8]. We shall obtain a discrete version of Lelieuvre's formula, and further give a discrete analogue of the affine Bäcklund transformation.

In 1980, S. S. Chern and C. L. Terng ([5]) studied the transformation of affine minimal surfaces by realizing them as the focal surfaces of a line congruence, and showed that there is an affine analogue of the classical Euclidean Bäcklund transformation. Whereas the classical one is defined for surfaces of constant negative Gaussian curvature, in the affine case the Bäcklund transformation is defined for affine minimal surfaces. They proved the following theorem.

Theorem 1.1 (Chern-Terng)**.** *Let $f\colon M \to \mathbb{R}^3$ be an affine minimal surface. Then, given any tangent vector v_0 in $T_{f(p_0)}\mathbb{R}^3$, there exists an affine minimal surface $\widehat{f}\colon M \to \mathbb{R}^3$ such that $\widehat{f}(p_0) - f(p_0) = v_0$ and that the correspondence of f to $\widehat{f}$ owns the following properties:*
(i) The vector $\widehat{f}(p) - f(p)$ is tangent to both of the immersions f and $\widehat{f}$, and
(ii) the Blaschke normal vectors $\xi(p)$ and $\widehat{\xi}(p)$ are parallel
for any point p in M.

Such a correspondence of immersions f to $\widehat{f}$ satisfying (i) and (ii) is called the *affine Bäcklund transformation*. Theorem 1.1 seemingly leads up to consecutive constructions of new affine minimal surfaces from a given trivial one, but S. Buyske [4] showed that this Bäcklund transformation can be simply represented by an involution and translation of the conormal map. By virtue of Buyske's expression and the discrete Lelieuvre formula, we can show that there exists a discrete analogue of the affine Bäcklund transformation (Theorem 4.3 and Theorem 6.2).

In the final section, we exhibit some fundamental examples of discete affine minimal surfaces. To provide such explicit examples, we shall study discrete harmonic polynomials in Section 7. By using Hirota's discrete power function, we give a complete classification of discrete harmonic polynomials of 2 variables (Theorem 7.2). The classification of discrete harmonic polynomial is of some interests regardless of geometry.

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2 Preliminaries

In this section, let us recall basic notation of affine differential geometry according to [9]. Let $f\colon M \to \mathbb{R}^3$ be an affine immersion provided with an equiaffine transversal vector field ξ, then the formulas of Gauss and Weingarten are as follows:

$$D_X(f_*Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \quad D_X\xi = -f_*(SX) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

Here ∇ is a torsion-free induced connection, h a symmetric covariant tensor field, and S a tensor field of type $(1,1)$. We assume that the affine fundamental form h has rank 2 and hence can be treated as a nondegenerate metric on M, which is traditionally called the *affine metric*. We shall use a fixed parallel volume element in $\mathbb{R}^3$ given by the determinant function det. An equiaffine transversal vector field ξ satisfying the volume condition

$$\det(f_*X, f_*Y, \xi) = \left|\det\begin{pmatrix} h(X,X) & h(X,Y) \\ h(Y,X) & h(Y,Y)\end{pmatrix}\right|^{1/2}$$

is called the *Blaschke normal field*, and the affine immersion with Blaschke normal field is called a *Blaschke immersion*. A Blaschke immersion is said to be *affine minimal* when $\mathrm{tr}S$ vanishes everywhere.

Here we review the notion of the conormal map, which helps us to treat with the affine Bäcklund transformation easily. The role of the conormal map will be later clarified in the following sections. The *conormal mapping* $\nu\colon M \to \mathbb{R}_3$ associated with a Blaschke immersion (f, ξ) is determined by the property

$$\langle \nu,\, f_*X\rangle = 0, \quad \langle \nu,\, \xi\rangle = 1$$

at each point of M, where $\mathbb{R}_3$ denotes the dual space of the underlying vector space for $\mathbb{R}^3$. Such a map ν is uniquely determined and is a centro-affine immersion. We write the formula of Gauss

$$D_X(\nu_*Y) = \nu_*(\overline{\nabla}_X Y) + \overline{h}(X,Y)(-\nu),$$

where $\overline{\nabla}$ is the induced affine connection on M by ν and $\overline{h}$ the affine fundamenteal form for ν. The pair $(\overline{\nabla}, \overline{h})$ is related to the Blaschke structure (∇, h, S) of the immersion f by

$$\overline{h}(X,Y) = h(SX, Y), \quad Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \overline{\nabla}_X Z).$$

These equations imply that $\overline{h}$ may be degenerate and $\overline{\nabla}$ is conjugate to ∇. We have the following formula for the Laplacian Δ relative to h applied to ν:

$$\Delta\nu + (\mathrm{tr}S)\,\nu = 0.$$

In particular, the conormal immersion associated with an affine minimal surface is harmonic.

By using the dual determinant function $\det^*$ on $\mathbb{R}_3$, which is defined as

$$\left|\det{}^*(\nu_*X, \nu_*Y, -\nu)\right| = \det(f_*X, f_*Y, \xi),$$

the exterior product on $\mathbb{R}_3$ is defined by the formula

$$\langle \nu, \nu_1 \times \nu_2\rangle = \det{}^*(\nu, \nu_1, \nu_2)$$

where $\nu, \nu_1, \nu_2 \in \mathbb{R}_3$.

3 Indefinite affine minimal surface

First, we treat affine minimal surfaces with indefinite affine metrics. Now let $f\colon M \to \mathbb{R}^3$ be an affine minimal surface with indefinite affine metric h. We call such a surface *indefinite affine minimal surface*. Choose an asymptotic coordinate system (x, y) defined on a (simply connected) region $\mathbb{D}$, then h is expressed as $h = 2\omega\, dxdy$. The formulas of Gauss and Weingarten are as follows:

$$f_{xx} = \frac{\omega_x}{\omega}f_x + \frac{a}{\omega}f_y, \quad f_{xy} = \omega\xi, \quad f_{yy} = \frac{b}{\omega}f_x + \frac{\omega_y}{\omega}f_y,$$

$$\xi_x = -sf_y, \quad \xi_y = -tf_x,$$

where differentiation of a vector relative to x, y is denoted by attaching these letters as subscript to the vector. The volume condition is

$$\det(f_x, f_y, \xi) = \omega.$$

The equations of Gauss and Codazzi are

(1) $$(\log\omega)_{xy} + ab\omega^{-2} = 0, \quad a_y + \omega^2 s = 0, \quad b_x + \omega^2 t = 0,$$

and

(2) $$(\omega t)_x - bs = 0, \quad (\omega s)_y - at = 0.$$

The formula of Gauss for the conormal map $\nu\colon \mathbb{D} \to \mathbb{R}_3$ is

$$\nu_{xx} = \frac{\omega_x}{\omega}\nu_x - \frac{a}{\omega}\nu_y + \omega s(-\nu), \quad \nu_{xy} = 0, \quad \nu_{yy} = -\frac{b}{\omega}\nu_x + \frac{\omega_y}{\omega}\nu_y + \omega t(-\nu),$$

which in patricular shows the conormal immersion is harmonic relative to the asymptotic coordinate system (x, y). This means that ν defines a translation surface, namely $\nu(x, y) = p(x) + q(y)$. We have so far defined the conormal map ν to a given immersion f. The original immersion f is described by *Lelieuvre's formula*

$$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \nu \times \nu_x dx - \nu \times \nu_y dy$$

in terms of ν and h. Here $\boldsymbol{x}_0$ is a base point. The affine Bäcklund transformation is described by the conormal $\nu(x, y) = p(x) + q(y)$ in the following way:

Theorem 3.1. *Let $f\colon \mathbb{D} \to \mathbb{R}^3$ be an indefinite affine minimal surface with conormal immersion $\nu(x, y) = p(x) + q(y)$, where (x, y) is an asymptotic coordinate system. Then the surface $\widehat{f}$ given by*

$$\widehat{f} = f - 2q \times p$$

is an indefinite affine minimal surface with conormal $\widehat{\nu}(x, y) = p(x) - q(y)$ such that the correspondence f to $\widehat{f}$ is an affine Bäcklund transformation.

4 Discrete indefinite affine minimal surface

We are now ready for discretizing indefinite affine minimal surfaces. We shall define a discrete indefinite affine minimal surface as a special asymptotic net. In the theory of discrete integrable geometry, it has been an intensive subject to discretize surfaces which allow asymptotic coordinate parametrizations.

By *discrete surface*, we mean a map $f\colon \varepsilon\mathbb{Z} \times \delta\mathbb{Z} \to \mathbb{R}^3$. We shall use the following difference operators:

$$\begin{aligned} \Delta_{+x} f(x, y) &= \frac{f(x+\varepsilon, y) - f(x, y)}{\varepsilon}, \\ \Delta_x f(x, y) &= \frac{f(x+\varepsilon/2, y) - f(x-\varepsilon/2, y)}{\varepsilon}, \\ \Delta_{-x} f(x, y) &= \frac{f(x, y) - f(x-\varepsilon, y)}{\varepsilon}, \end{aligned}$$

where ε denotes a difference interval with respect to $x \in \varepsilon\mathbb{Z}$.

Definition 4.1. A map $f\colon \varepsilon\mathbb{Z} \times \delta\mathbb{Z} \to \mathbb{R}^3$ is called a *discrete indefinite affine minimal surface* if there exist a map $\xi\colon \varepsilon\mathbb{Z} \times \delta\mathbb{Z} \to \mathbb{R}^3$ and five functions $a, b, \omega, s, t\colon \varepsilon\mathbb{Z} \times \delta\mathbb{Z} \to \mathbb{R}$ which satisfy the following difference systems (3)–(7):
Discrete Gauss formula:

(3) $$\Delta_x^2 f(x, y) = \frac{\Delta_{-x}\omega(x, y)}{\omega(x, y)} \Delta_{+x} f(x, y) + \frac{a(x, y)}{\omega(x, y)} \Delta_{+y} f(x, y),$$

(4) $$\Delta_{+x}\Delta_{+y} f(x, y) = \omega(x, y)\xi(x, y),$$

(5) $$\Delta_y^2 f(x, y) = \frac{b(x, y)}{\omega(x, y)} \Delta_{+x} f(x, y) + \frac{\Delta_{-y}\omega(x, y)}{\omega(x, y)} \Delta_{+y} f(x, y);$$

Discrete Weingarten formula:

(6) $$\Delta_{-x}\xi(x, y) = -s(x, y)\Delta_{+y} f(x, y), \quad \Delta_{-y}\xi(x, y) = -t(x, y)\Delta_{+x} f(x, y);$$

Volume condition:

(7) $$\omega(x, y) = \det(\Delta_{+x} f(x, y), \Delta_{+y} f(x, y), \xi(x, y)).$$

The equations (3) and (5) imply that f makes an asymptotic net.

Even though the defining relations (3)–(7) have been satisfied identically, there exist further constraints on the coefficients of the discrete Gauss and Weingarten formulas due to compatibility. The equations of Gauss

$$\begin{aligned}\Delta_{+y}\left(\Delta_{-x}\Delta_{+x}f(x,y)\right) &= \Delta_{-x}\left(\Delta_{+y}\Delta_{+x}f(x,y)\right),\\ \Delta_{+x}\left(\Delta_{-y}\Delta_{+y}f(x,y)\right) &= \Delta_{-y}\left(\Delta_{+x}\Delta_{+y}f(x,y)\right)\end{aligned}$$

reduce to the following three equations:

$$\begin{gathered}\omega(x,y)\omega(x-\varepsilon,y-\delta)-\omega(x-\varepsilon,y)\omega(x,y-\delta)+\varepsilon\delta a(x,y)b(x,y)=0,\\ \Delta_{+y}a(x,y)+\omega(x-\varepsilon,y)\omega(x,y)s(x,y)=0,\\ \Delta_{+x}b(x,y)+\omega(x,y-\delta)\omega(x,y)t(x,y)=0.\end{gathered}$$

The equation of Codazzi $\Delta_{-x}\Delta_{-y}\xi(x,y)=\Delta_{-y}\Delta_{-x}\xi(x,y)$ becomes

$$\begin{aligned}\Delta_{-x}\left(\omega(x,y)t(x,y)\right)-b(x,y)s(x,y-\delta) &= 0,\\ \Delta_{-y}\left(\omega(x,y)s(x,y)\right)-a(x,y)t(x-\varepsilon,y) &= 0.\end{aligned}$$

From these equations, one can easily check that the smooth equations (1) and (2) would be recovered in a small limit of ε and δ.

Remark 4.2. If ξ is a constant mapping, the map f is especially said to be a *discrete indefinite improper affine sphere* introduced by the author and Urakawa [8]. A discrete indefinite improper affine sphere is a geometric model of Hirota's discrete Liouville equation.

We define a map $\nu\colon \varepsilon\mathbb{Z}\times\delta\mathbb{Z}\to\mathbb{R}_3$, called the *discrete conormal map* associated with f, as

$$\nu(x,y)=\frac{1}{\omega(x,y)}\Delta_{+x}f(x,y)\times\Delta_{+y}f(x,y)$$

and hence obtain the *discrete Lelieuvre formula*

$$\Delta_{+x}f(x,y)=\nu(x,y)\times\Delta_{+x}\nu(x,y),\quad \Delta_{+y}f(x,y)=\Delta_{+y}\nu(x,y)\times\nu(x,y).$$

The formula of Gauss for ν is

$$\begin{aligned}\Delta_x^2\nu(x,y) &= \frac{\Delta_{-x}\omega(x,y)}{\omega(x,y)}\Delta_{+x}\nu(x,y)-\frac{a(x,y)}{\omega(x,y)}\Delta_{+y}\nu(x,y)\\ &\quad +\omega(x-\varepsilon,y)s(x,y)(-\nu(x,y)),\\ \Delta_{+x}\Delta_{+y}\nu(x,y) &= 0,\\ \Delta_y^2\nu(x,y) &= -\frac{b(x,y)}{\omega(x,y)}\Delta_{+x}\nu(x,y)+\frac{\Delta_{-y}\omega(x,y)}{\omega(x,y)}\Delta_{+y}\nu(x,y)\\ &\quad +\omega(x,y-\delta)t(x,y)(-\nu(x,y)).\end{aligned}$$

In particular the second equation shows that the discrete conormal map ν is harmonic as in the smooth case, therefore we can set $\nu(x,y)=p(x)+q(y)$.

Theorem 4.3. *Let $f\colon \varepsilon\mathbb{Z}\times\delta\mathbb{Z}\to\mathbb{R}^3$ be a discrete indefinite affine minimal surface. Then the mapping $\widehat{f}\colon \varepsilon\mathbb{Z}\times\delta\mathbb{Z}\to\mathbb{R}^3$ given by*

$$\widehat{f}(x,y)=f(x,y)-2q(y)\times p(x)$$

defines another discrete indefinite affine minimal surface, where $p(x)+q(y)$ is the discrete conormal map associated with f. The correspondence $f(x,y)\mapsto\widehat{f}(x,y)$ has the following property: The vectors $\xi(x,y)$ and $\widehat{\xi}(x,y)$ are parallel for any point $(x,y)\in\varepsilon\mathbb{Z}\times\delta\mathbb{Z}$. Further the vector $\widehat{f}(x,y)-f(x,y)$ is 'tangent' to both of the maps f and $\widehat{f}$ for any point (x,y), namely we have that

$$\begin{aligned}\det\Big(\Delta_{+x}f(x,y),\ \Delta_{+y}f(x,y),\ \widehat{f}(x,y)-f(x,y)\Big) &= 0,\\ \det\Big(\Delta_{+x}\widehat{f}(x,y),\ \Delta_{+y}\widehat{f}(x,y),\ \widehat{f}(x,y)-f(x,y)\Big) &= 0.\end{aligned}$$

Proof. We set $\widehat{\nu}(x,y) = p(x,y) - q(x,y)$ and obtain

$$\Delta_{+x}\widehat{f}(x,y) = \widehat{\nu}(x,y) \times \Delta_{+x}\widehat{\nu}(x,y), \quad \Delta_{+y}\widehat{f}(x,y) = \Delta_{+y}\widehat{\nu}(x,y) \times \widehat{\nu}(x,y),$$

which means that the discrete Lelieuvre formula is also valid for $\widehat{f}$. The map $\widehat{f}$ is an indefinite affine minimal surface, because $\widehat{f}$ satisfies the relations (3)–(7) with the following functions:

$$\begin{aligned} \widehat{\omega}(x,y) &= -\det\left(\widehat{\nu}(x,y), \Delta_{+x}\widehat{\nu}(x,y), \Delta_{+y}\widehat{\nu}(x,y)\right), \\ \widehat{a}(x,y) &= \det\left(\widehat{\nu}(x,y), \Delta_{+x}\widehat{\nu}(x,y), \Delta_x^2\widehat{\nu}(x,y)\right), \\ \widehat{b}(x,y) &= \det\left(\widehat{\nu}(x,y), \Delta_{+y}\widehat{\nu}(x,y), \Delta_y^2\widehat{\nu}(x,y)\right). \end{aligned}$$

In fact, the Gauss formula for $\widehat{\nu}$ proves that

$$\begin{aligned} \Delta_x^2\widehat{f}(x,y) &= \Delta_{-x}\left\{\widehat{\nu}(x,y) \times \Delta_{+x}\widehat{\nu}(x,y)\right\} \\ &= \Delta_{-x}\widehat{\nu}(x,y) \times \Delta_{+x}\widehat{\nu}(x-\varepsilon,y) + \widehat{\nu}(x,y) \times \Delta_{-x}\Delta_{+x}\widehat{\nu}(x,y) \\ &= \widehat{\nu}(x,y) \times \Delta_x^2\widehat{\nu}(x,y) \\ &= \frac{\Delta_{-x}\widehat{\omega}(x,y)}{\widehat{\omega}(x,y)}\widehat{\nu}(x,y) \times \Delta_{+x}\widehat{\nu}(x,y) - \frac{\widehat{a}(x,y)}{\widehat{\omega}(x,y)}\widehat{\nu}(x,y) \times \Delta_{+y}\widehat{\nu}(x,y) \\ &= \frac{\Delta_{-x}\widehat{\omega}(x,y)}{\widehat{\omega}(x,y)}\Delta_{+x}\widehat{f}(x,y) + \frac{\widehat{a}(x,y)}{\widehat{\omega}(x,y)}\Delta_{+y}\widehat{f}(x,y). \end{aligned}$$

Similarly we obtain

$$\Delta_{+x}\Delta_{+y}\widehat{f}(x,y) = \widehat{\omega}(x,y)\ \widehat{\xi}(x,y),$$

where $\widehat{\xi}$ is defined by $\widehat{\xi} = (-\omega/\widehat{\omega})\,\xi$. □

5 Definite affine minimal surface

We next consider an affine minimal surface with definite metric. Similarly in the case of indefinite metric, we first review the continuous case. Assume now that the affine metric h of an affine minimal surface $f\colon (\mathbb{D},(x,y)) \to \mathbb{R}^3$ is positive definite. We briefly say that f is a *definite affine minimal surface.* We choose an isothermal coordinate system (x,y) with respect to h so that $h = 2\omega\, dzd\bar{z}$, where we introduce the complex coordinate $z = x + \sqrt{-1}y, \bar{z} = x - \sqrt{-1}y$. We have the formulas of Gauss and Weingarnten

$$f_{zz} = \frac{\omega_z}{\omega}f_z + \frac{c}{\omega}f_{\bar{z}}, \quad f_{z\bar{z}} = \omega\xi, \quad \xi_z = -uf_{\bar{z}}$$

and the volume condition

$$\omega = -\sqrt{-1}\det(f_z, f_{\bar{z}}, \xi).$$

These coefficient functions c, u and ω must satisfy the compatibility conditions

$$(\log\omega)_{z\bar{z}} + |c|^2\omega^{-2} = 0, \quad c_{\bar{z}} + u\omega^2 = 0, \quad (\bar{u}\omega)_z - \bar{c}u = 0.$$

The formula of Gauss for the conormal map $\nu\colon \mathbb{D} \to \mathbb{R}_3$ is

$$\nu_{zz} = \frac{\omega_z}{\omega}\nu_z - \frac{c}{\omega}\nu_{\bar{z}} + \omega u(-\nu), \quad \nu_{z\bar{z}} = 0.$$

Thus the map ν is an $\mathbb{R}_3$-valued harmonic function, hence there exists a $\mathbb{C}_3$-valued holomorphic map w whose imaginary part is ν. Any definite affine minimal surface has locally an integral representation

$$f(z,\bar{z}) = -\frac{\sqrt{-1}}{4}\left\{w \times \bar{w} + \int^z w \times dw - \bar{w} \times d\bar{w}\right\}$$

up to a constant vector. This integral formula is called the *affine Weierstrass formula*, which is a simple consequence of *Lelieuvre's formula* of definite version:

(8) $$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \nu \times \nu_y dx - \nu \times \nu_x dy.$$

The affine Bäcklund transformation is described in the following way.

Theorem 5.1. *Consider a definite affine minimal surface $f\colon \mathbb{D} \to \mathbb{R}^3$ with conormal $\nu = \Im(w)$ for some holomorphic map $w\colon \mathbb{D} \to \mathbb{C}_3$. The surface $\widehat{f}$ given by*

$$\widehat{f} = f + \frac{\sqrt{-1}}{2} w \times \bar{w}$$

is a definite affine minimal surface with conormal $\widehat{\nu} = \Im(\widehat{w})$, where $\widehat{w} = \sqrt{-1}w$, such that the correspondence f to $-\widehat{f}$ is an affine Bäcklund transformation.

6 Discrete definite affine minimal surface

In view of Lelieuvre's formula (8), it is convenient to make the following definition.

Definition 6.1. A map $f\colon \varepsilon\mathbb{Z} \times \delta\mathbb{Z} \to \mathbb{R}^3$ is called a *discrete definite affine minimal surface* if it is determined by the *discrete Lelieuvre formula*

$$\begin{aligned} \Delta_{-x} f(x,y) &= \nu(x, y+\delta) \times \Delta_{+y}\nu(x,y), \\ \Delta_{-y} f(x,y) &= \Delta_{+x}\nu(x,y) \times \nu(x+\varepsilon, y), \end{aligned}$$

where $\nu\colon \varepsilon\mathbb{Z} \times \delta\mathbb{Z} \to \mathbb{R}_3$ is a discrete harmonic map, namely ν satisfies that

(9) $$\Delta_x^2 \nu(x,y) + \Delta_y^2 \nu(x,y) = 0.$$

Let (f, ν) be a discrete definite affine minimal surface. Because ν is discrete harmonic (9), we can set

$$\begin{aligned} \Delta_x^2 \nu(x,y) &= \alpha(x,y)\Delta_{-x}\nu(x,y) - \beta(x,y)\Delta_{-y}\nu(x,y) - \gamma(x,y)\nu(x,y), \\ \Delta_{-x}\Delta_{-y}\nu(x,y) &= p(x,y)\Delta_{-x}\nu(x,y) + q(x,y)\Delta_{-y}\nu(x,y) + r(x,y)\nu(x,y), \\ \Delta_y^2 \nu(x,y) &= -\alpha(x,y)\Delta_{-x}\nu(x,y) + \beta(x,y)\Delta_{-y}\nu(x,y) + \gamma(x,y)\nu(x,y). \end{aligned}$$

Then, the discrete Gauss formula for f is as follows:

$$\begin{aligned} \Delta_x^2 f(x,y) &= q(x+\varepsilon, y+\delta)\Delta_{+x} f(x,y) \\ &\quad - \{p(x+\varepsilon, y+\delta) + \varepsilon r(x+\varepsilon, y+\delta)\}\,\Delta_{+y} f(x,y) \\ &\quad\quad +2\,\{1 - \varepsilon q(x+\varepsilon, y+\delta)\}\,\omega(x+\varepsilon, y+\delta)\xi(x,y), \\ \Delta_{+x}\Delta_{+y} f(x,y) &= \beta(x+\varepsilon, y+\delta)\Delta_{+x} f(x,y) \\ &\quad +\alpha(x+\varepsilon, y+\delta)\Delta_{+y} f(x,y), \\ \Delta_y^2 f(x,y) &= -\{q(x+\varepsilon, y+\delta) + \delta r(x+\varepsilon, y+\delta)\}\,\Delta_{+x} f(x,y) \\ &\quad +p(x+\varepsilon, y+\delta)\Delta_{+y} f(x,y) \\ &\quad\quad +2\,\{1 - \delta p(x+\varepsilon, y+\delta)\}\,\omega(x+\varepsilon, y+\delta)\xi(x,y). \end{aligned}$$

Here $\xi\colon \varepsilon\mathbb{Z} \times \delta\mathbb{Z} \to \mathbb{R}^3$ is defined by

$$\xi(x,y) = \frac{1}{2\omega(x+\varepsilon, y+\delta)} \Delta_{-x}\nu(x+\varepsilon, y+\delta) \times \Delta_{-y}\nu(x+\varepsilon, y+\delta),$$

where

$$2\omega(x,y) = \det(\Delta_{-x}\nu(x,y), \Delta_{-y}\nu(x,y), \nu(x,y)).$$

The formula of Weingarten is

$$\begin{aligned} \Delta_{+x}\xi(x,y) &= -\frac{\gamma(x+\varepsilon, y+\delta) - \varepsilon r(x+2\varepsilon, y+\delta)\beta(x+\varepsilon, y+\delta)}{2\omega(x+\varepsilon, y+\delta)\{1 + \varepsilon\alpha(x+\varepsilon, y+\delta)\}}\Delta_{+x} f(x,y) \\ &\quad +\frac{r(x+2\varepsilon, y+\delta)}{2\omega(x+\varepsilon, y+\delta)}\Delta_{+y} f(x,y), \\ \Delta_{+y}\xi(x,y) &= \frac{r(x+\varepsilon, y+2\delta)}{2\omega(x+\varepsilon, y+\delta)}\Delta_{+x} f(x,y) \\ &\quad +\frac{\gamma(x+\varepsilon, y+\delta) + \delta r(x+\varepsilon, y+2\delta)\alpha(x+\varepsilon, y+\delta)}{2\omega(x+\varepsilon, y+\delta)\{1 + \delta\beta(x+\varepsilon, y+\delta)\}}\Delta_{+y} f(x,y). \end{aligned}$$

Thus, the pair (f, ξ) obeys rather complicated difference systems above, which provides us an explanation of our utilizing Lelieuvre's formula for discretizing definite affine minimal surfaces.

In any case, those coefficients must satisfy the compatibility conditions (10)–(15):

$$
\begin{aligned}
(10)\quad &\frac{1-\delta p(x,y)}{1+\delta\beta(x,y-\delta)}\left(\Delta_{-y}\alpha(x,y)+\alpha(x,y)\beta(x,y-\delta)\right)\\
&= \frac{1+\varepsilon\alpha(x,y)}{1-\varepsilon q(x+\varepsilon,y)}\left(\Delta_{+x}p(x,y)+p(x,y)q(x+\varepsilon,y)+r(x+\varepsilon,y)\right),
\end{aligned}
$$

$$
\begin{aligned}
(11)\quad &\frac{1}{1+\delta\beta(x,y-\delta)}\left(-\Delta_{-y}\beta(x,y)-\beta(x,y)\beta(x,y-\delta)+\alpha(x,y-\delta)q(x,y)-\gamma(x,y-\delta)\right)\\
&= \frac{1}{1-\varepsilon q(x+\varepsilon,y)}\left(\Delta_{+x}q(x,y)+q(x,y)q(x+\varepsilon,y)-\left(p(x+\varepsilon,y)+\varepsilon r(x+\varepsilon,y)\right)\beta(x,y)\right),
\end{aligned}
$$

$$
\begin{aligned}
(12)\quad &\frac{1}{1+\delta\beta(x,y-\delta)}\left(-\Delta_{-y}\gamma(x,y)-\beta(x,y-\delta)\gamma(x,y)+\alpha(x,y-\delta)r(x,y)\right)\\
&= \frac{1}{1-\varepsilon q(x+\varepsilon,y)}\left(\Delta_{+x}r(x,y)+q(x+\varepsilon,y)r(x,y)-\left(p(x+\varepsilon,y)+\varepsilon r(x+\varepsilon,y)\right)\gamma(x,y)\right),
\end{aligned}
$$

$$
\begin{aligned}
(13)\quad &\frac{1}{1+\varepsilon\alpha(x-\varepsilon,y)}\left(-\Delta_{-x}\alpha(x,y)-\alpha(x,y)\alpha(x-\varepsilon,y)+\beta(x-\varepsilon,y)p(x,y)+\gamma(x-\varepsilon,y)\right)\\
&= \frac{1}{1-\delta p(x,y+\delta)}\left(\Delta_{+y}p(x,y)+p(x,y)p(x,y+\delta)-\left(q(x,y+\delta)+\delta r(x,y+\delta)\right)\alpha(x,y)\right),
\end{aligned}
$$

$$
\begin{aligned}
(14)\quad &\frac{1-\varepsilon q(x,y)}{1+\varepsilon\alpha(x-\varepsilon,y)}\left(\Delta_{-x}\beta(x,y)+\alpha(x-\varepsilon,y)\beta(x,y)\right)\\
&= \frac{1+\delta\beta(x,y)}{1-\delta p(x,y+\delta)}\left(\Delta_{+y}q(x,y)+p(x,y+\delta)q(x,y)+r(x,y+\delta)\right),
\end{aligned}
$$

$$
\begin{aligned}
(15)\quad &\frac{1}{1+\varepsilon\alpha(x-\varepsilon,y)}\left(\Delta_{-x}\gamma(x,y)+\alpha(x-\varepsilon,y)\gamma(x,y)+\beta(x-\varepsilon,y)r(x,y)\right)\\
&= \frac{1}{1-\delta p(x,y+\delta)}\left(\Delta_{+y}r(x,y)+p(x,y+\delta)r(x,y)+\left(q(x,y+\delta)+\delta r(x,y+\delta)\right)\gamma(x,y)\right).
\end{aligned}
$$

To show that (10) and (14) are equivalent, we introduce two functions a and b by

$$
\begin{aligned}
2a(x,y) &= \omega(x,y)q(x,y)-\omega(x-\varepsilon,y)\alpha(x-\varepsilon,y),\\
2b(x,y) &= \omega(x,y)p(x,y)-\omega(x,y-\delta)\beta(x,y-\delta),
\end{aligned}
$$

which result in the following expressions:

$$
\begin{aligned}
\alpha(x-\varepsilon,y) &= \frac{\Delta_{-x}\omega(x,y)-2a(x,y)}{2\omega(x-\varepsilon,y)},\\
\beta(x,y-\delta) &= \frac{\Delta_{-y}\omega(x,y)-2b(x,y)}{2\omega(x,y-\delta)},\\
p(x,y) &= \frac{\Delta_{-y}\omega(x,y)+2b(x,y)}{2\omega(x,y)},\\
q(x,y) &= \frac{\Delta_{-x}\omega(x,y)+2a(x,y)}{2\omega(x,y)}.
\end{aligned}
$$

On using these expressions, we see that equations (10) and (14) both reduces to the equation

$$
(16)\qquad \Delta_{-y}a(x,y)+\Delta_{-x}b(x,y)+\omega(x,y)r(x,y)=0,
$$

and are hence equivalent. Thus, the compatibility conditions are (11), (12), (13), (15) and (16).

We define a map $\eta\colon \varepsilon\mathbb{Z}\times\delta\mathbb{Z}\to\mathbb{R}_3$ by the relation

$$
(17)\qquad \Delta_{+x}\eta(x,y)=\Delta_{-y}\nu(x,y),\quad \Delta_{+y}\eta(x,y)=-\Delta_{-x}\nu(x,y).
$$

We have

Theorem 6.2. *Let $f\colon \varepsilon\mathbb{Z}\times\delta\mathbb{Z}\to\mathbb{R}^3$ be a discrete definite affine minimal surface. Then the mapping $\widehat{f}\colon \varepsilon\mathbb{Z}\times\delta\mathbb{Z}\to\mathbb{R}^3$ given by*

$$\widehat{f}(x,y) = -f(x-\varepsilon, y-\delta) - \eta(x,y)\times\nu(x,y)$$

defines another discrete definite affine minimal surface. The correspondence $f(x,y)\mapsto -\widehat{f}(x+\varepsilon, y+\delta)$ has the following properties: The vectors $\xi(x,y)$ and $\widehat{\xi}(x+\varepsilon,y+\delta)$ are parallel for any point $(x,y)\in\varepsilon\mathbb{Z}\times\delta\mathbb{Z}$. Further the vector $\widehat{f}(x+\varepsilon,y+\delta)+f(x,y)$ is 'tangent' to both of the maps f and $\widehat{f}$ for any point (x,y), that is, we have that

$$\begin{aligned}\det\Big(\Delta_{+x}f(x,y),\ \Delta_{+y}f(x,y),\ \widehat{f}(x+\varepsilon,y+\delta)+f(x,y)\Big) &= 0,\\ \det\Big(\Delta_{+x}\widehat{f}(x,y),\ \Delta_{+y}\widehat{f}(x,y),\ \widehat{f}(x,y)+f(x-\varepsilon,y-\delta)\Big) &= 0.\end{aligned}$$

Proof. The map $\widehat{f}$ is recovered by a discrete harmonic map η as

$$\Delta_{-x}\widehat{f}(x,y) = \eta(x,y+\delta)\times\Delta_{+y}\eta(x,y), \quad \Delta_{-y}\widehat{f}(x,y) = -\Delta_{+x}\eta(x,y)\times\eta(x+\varepsilon,y).$$

Then, by definition, $\widehat{f}$ makes another discrete definite affine minimal surface. □

7 Discrete harmonic polynomial

To provide concrete examples of discrete affine minimal surfaces, we investigate discrete harmonic polynomials in this section. In 1949 H. A. Heilbronn [6] introduced the notion of discrete harmonic functions, however he did not classify all discrete harmonic polynomials. His method of finding discrete harmonic polynomials has some ambiguities to determine such polynomials. To exclude such ambiguities and give a complete classification, we use Hirota's mean power function $x^{(k)}$.

Definition 7.1 (Hirota [7])**.**

$$x^{(k)} = \begin{cases} \dfrac{1}{x^{(-k)}} & \text{for } k<0,\\ 1 & \text{for } k=0,\\ \displaystyle\prod_{i=1}^{k}\left(x+\left(\frac{k+1}{2}-i\right)\varepsilon\right) & \text{for } k>0.\end{cases}$$

It is remarkable that the mean power function $x^{(k)}$ admits Leibniz' rule with respect to the center difference operator Δ_x. Namely we have that $\Delta_x x^{(k)} = kx^{(k-1)}$ for all integer k.

Now we give the classification of discrete harmonic polynomials of 2 variables.

Theorem 7.2. *Let $\psi_d(x,y)$ be a discrete harmonic polynomial of degree $d>0$. Then $\psi_d(x,y)$ is a linear combination of the discrete harmonic polynomials $\varphi_d(x,y)$ and $\phi_d(x,y)$ given by*

$$\begin{aligned}\varphi_d(x,y) &= \sum_{i=0}^{[d/2]}(-1)^i\binom{d}{2i}x^{(d-2i)}y^{(2i)},\\ \phi_d(x,y) &= \sum_{i=0}^{[(d-1)/2]}(-1)^i\binom{d}{2i+1}x^{(d-2i-1)}y^{(2i+1)}.\end{aligned}$$

Proof. Let $\psi_d(x,y)=\sum_{k=0}^{d}c_k x^{(d-k)}y^{(k)}$ be a discrete harmonic polynomial of degree d. We shall determine the coefficients c_k. Because ψ_d is discrete harmonic,

$$\begin{aligned}0 &= (\Delta_x^2+\Delta_y^2)\psi_d(x,y)\\ &= \sum_{k=0}^{d-2}\Big((d-k-1)(d-k)c_k+(k+2)(k+1)c_{k+2}\Big)x^{(d-k-2)}y^{(k)}.\end{aligned}$$

Hence $(d-k-1)(d-k)c_k + (k+2)(k+1)c_{k+2} = 0$ for all k. Therefore c_k take the following form

$$\begin{cases} c_{2i} = (-1)^i \binom{d}{2i} c_0 & \text{for} \quad 0 \le i \le \left[\dfrac{d}{2}\right], \\ c_{2i+1} = (-1)^i \binom{d}{2i+1} \dfrac{c_1}{d} & \text{for} \quad 0 \le i \le \left[\dfrac{d-1}{2}\right]. \end{cases}$$

Consequently, a discrete harmonic polynomial ψ_d is expressed as

$$\begin{aligned} \psi_d(x,y) &= c_0 \sum_{i=0}^{[d/2]} (-1)^i \binom{d}{2i} x^{(d-2i)} y^{(2i)} + \frac{c_1}{d} \sum_{i=0}^{[(d-1)/2]} (-1)^i \binom{d}{2i+1} x^{(d-2i-1)} y^{(2i+1)} \\ &= c_0 \varphi_d(x,y) + \frac{c_1}{d} \phi_d(x,y) \end{aligned}$$

where c_0 and c_1 are arbitrary constants. □

8 Examples

We illustrate two examples, one is indefinite and the other definite. We shall use the following power functions: The descending power function

$$x^{\underline{k}} = \begin{cases} \displaystyle\prod_{i=1}^{-k} \frac{1}{x+i\varepsilon} & \text{for } k < 0, \\ 1 & \text{for } k = 0, \\ \displaystyle\prod_{i=1}^{k} (x-(i-1)\varepsilon) & \text{for } k > 0, \end{cases}$$

and the ascending power function

$$x^{\overline{k}} = \begin{cases} \displaystyle\prod_{i=1}^{-k} \frac{1}{x-i\varepsilon} & \text{for } k < 0, \\ 1 & \text{for } k = 0, \\ \displaystyle\prod_{i=1}^{k} (x+(i-1)\varepsilon) & \text{for } k > 0. \end{cases}$$

The descending (resp. ascending) power function admits Leibniz' rule with respect to the forward (resp. backward) difference operator. Namely we have have that $\Delta_{+x} x^{\underline{k}} = k x^{\underline{k-1}}$ and $\Delta_{-x} x^{\overline{k}} = k x^{\overline{k-1}}$ for all integer k.

Example 8.1 (discrete Enneper surface)**.** Let α, β and γ be constants. The discrete conormal map $\nu(x,y) = p(x) + q(y)$ given by

$$p(x) = 9 \begin{pmatrix} -2(x-\varepsilon) \\ 0 \\ x^{\underline{2}} - 1/2 \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad q(y) = 9 \begin{pmatrix} 0 \\ -2(y-\delta) \\ y^{\underline{2}} - 1/2 \end{pmatrix} - \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

satisfies the formula of Gauss with the following functions:

$$a(x,y) = -2(y-\delta), \quad b(x,y) = -2(x-\varepsilon),$$

$$s(x,y) = \frac{2}{\omega(x,y)\omega(x-\varepsilon,y)}, \quad t(x,y) = \frac{2}{\omega(x,y)\omega(x,y-\delta)},$$

and

$$\omega(x,y) = 1 + x^{\underline{2}} + y^{\underline{2}}.$$

Then, by the discrete Lelieuvre formula, the original discrete surface f is recovered as

$$f(x,y) = \begin{pmatrix} y^{\underline{3}} - 3x^{\underline{2}}(y-\delta) + 3(y-\delta) \\ x^{\underline{3}} - 3(x-\varepsilon)y^{\underline{2}} + 3(x-\varepsilon) \\ -6(x-\varepsilon)(y-\delta) \end{pmatrix},$$

which is called a *discrete Enneper surface*. The defining relations (3)–(7) are all satisfied with the discrete affine normal

$$\xi(x,y) = -\frac{6}{\omega(x,y)} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Therefore f is a discrete indefinite affine minimal surface. The following map

$$\begin{aligned} \widehat{f}(x,y) \quad &= \quad \begin{pmatrix} y^{\underline{3}} + 321x^{\underline{2}}(y-\delta) - 159(y-\delta) \\ x^{\underline{3}} + 321(x-\varepsilon)y^{\underline{2}} - 159(x-\varepsilon) \\ 642(x-\varepsilon)(y-\delta) \end{pmatrix} \\ &\quad -18\alpha \begin{pmatrix} 0 \\ x^{\underline{2}} + y^{\underline{2}} - 1 \\ 2(y-\delta) \end{pmatrix} + 18\beta \begin{pmatrix} x^{\underline{2}} + y^{\underline{2}} - 1 \\ 0 \\ 2(x-\varepsilon) \end{pmatrix} + 36\gamma \begin{pmatrix} y-\delta \\ -(x-\varepsilon) \\ 0 \end{pmatrix} \end{aligned}$$

is again a discrete indefinite affine minimal surface. In fact $\widehat{f}$ satisfies the formulas of Gauss and Weingarten (3)–(6) with the coefficient functions

$$\widehat{a}(x,y) = 648\left(9(y-\delta)+\beta\right), \quad \widehat{b}(x,y) = 648\left(9(x-\varepsilon)-\alpha\right),$$

$$\widehat{s}(x,y) = \frac{2}{\widehat{\omega}(x,y)\widehat{\omega}(x-\varepsilon,y)}, \quad \widehat{t}(x,y) = \frac{2}{\widehat{\omega}(x,y)\widehat{\omega}(x,y-\delta)}$$

and

$$\widehat{\omega}(x,y) = -324\left(9x^{\underline{2}} - 9y^{\underline{2}} + 2(\alpha x - \beta y - \gamma)\right).$$

The discrete affine normal is $\widehat{\xi} = -\left(\omega/\widehat{\omega}\right)\xi$. The discrete conormal map is

$$\widehat{\nu}(x,y) = 9 \begin{pmatrix} -2(x-\varepsilon) \\ 2(y-\delta) \\ x^{\underline{2}} - y^{\underline{2}} \end{pmatrix} + 2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Example 8.2. Let $\psi_2(x,y)$ be a discrete harmonic polynomial of degree 2 given in Section 7. We have that

$$\psi_2(x,y) = c_0\varphi_2(x,y) + \frac{c_1}{2}\phi_2(x,y), \quad \varphi_2(x,y) = x^{(2)} - y^{(2)}, \quad \phi_2(x,y) = 2xy,$$

where c_0 and c_1 are constants. The discete conormal map

$$\nu(x,y) = \begin{pmatrix} x \\ y \\ \psi_2(x,y) \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

is discrete harmonic (9), where a_1, a_2 and a_3 are arbitrary constants. Hence, the mapping f recovered by ν as

$$\begin{aligned} f(x,y) \quad &= \quad \frac{1}{2} \begin{pmatrix} -2a_3x \\ -2a_3y \\ x^{\overline{2}} + y^{\overline{2}} + 2a_1x + 2a_2y \end{pmatrix} \\ &\quad + \frac{c_0}{6} \begin{pmatrix} -2x^{(3)} - 6xy^{\overline{2}} - 6a_2\big(x(y+\delta) + (x+\varepsilon)y\big) - 3\varepsilon\big(x^{\overline{2}} + y^{\overline{2}}\big) + (3/2)\left(\varepsilon^2 - \delta^2\right)x \\ 2y^{(3)} + 6x^{\overline{2}}y + 6a_1\big(x(y+\delta) + (x+\varepsilon)y\big) + 3\delta\big(x^{\overline{2}} + y^{\overline{2}}\big) + (3/2)\left(\varepsilon^2 - \delta^2\right)y \\ 0 \end{pmatrix} \\ &\quad + \frac{c_1}{6} \begin{pmatrix} -2y^{(3)} + 3a_2\big(x^{\overline{2}} - y^{\overline{2}}\big) - 3\delta y^{\overline{2}} \\ -2x^{(3)} - 3a_1\big(x^{\overline{2}} - y^{\overline{2}}\big) - 3\varepsilon x^{\overline{2}} \\ 0 \end{pmatrix} \end{aligned}$$

is, by definition, a discrete definite affine minimal surface. The compatibility conditions (11), (12), (13), (15) and (16) are now all trivial. The relation (17) determines another discrete harmonic map

$$\eta(x,y) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} - c_0 \begin{pmatrix} 0 \\ 0 \\ x(y-\delta) + (x-\varepsilon)y \end{pmatrix} + \frac{c_1}{2} \begin{pmatrix} 0 \\ 0 \\ x^{\underline{2}} - y^{\underline{2}} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

where b_1, b_2 and b_3 are arbitrary constants. Then,

$$\begin{aligned}\widehat{f}(x,y) &= \frac{1}{2}\begin{pmatrix} 2b_3y - 2a_3b_2 + 2a_2b_3 - 2\varepsilon a_3 \\ -2b_3x - 2a_1b_3 + 2a_3b_1 - 2\delta a_3 \\ x^{\overline{2}} + y^{\overline{2}} + 2b_2x - 2b_1y - 2a_2b_1 + 2a_1b_2 + 2\varepsilon a_1 + 2\delta a_2 \end{pmatrix} \\ &\quad - \frac{c_0}{6}\begin{pmatrix} 4x^{(3)} + 6b_2\left(x^{(2)} - y^{(2)}\right) + 3\varepsilon\left(x^{\overline{2}} - y^{\overline{2}} - (\varepsilon^2 - \delta^2)/2\right) \\ -4y^{(3)} - 6b_1\left(x^{(2)} - y^{(2)}\right) + 3\delta\left(x^{\overline{2}} - y^{\overline{2}} - (\varepsilon^2 - \delta^2)/2\right) \\ 0 \end{pmatrix} \\ &\quad - \frac{c_1}{6}\begin{pmatrix} y^{(3)} + 3x^{\overline{2}}y + 6b_2xy \\ x^{(3)} + 3xy^{\overline{2}} - 6b_1xy \\ 0 \end{pmatrix}\end{aligned}$$

is again a discrete definite affine minimal surface.

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