Noncommutative Integration in Partial O*-algebras

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Abstract

To investigate a construction of partial O*-algebras, we consider trace representations of weights and unbounded conditional expectations for partial O*-algebras.

1 Introduction

Algebras of unbounded operators called O*-algebras and partial O*-algebras have been studying from the pure mathematical situations and the physical applications. In this paper we investigate noncommutative integration in partial O*-algebras. In integral theory and probability theory, the Radon-Nikodým theorem, the Lebesgue decomposition theorem and conditional expectations play a fundamental role. Their noncommutative analogues in von Neumann algebras have been studied in [28, 29]. A typical feature of integral in von Neumann algebras is that the observables permitted are usually bounded and some finiteness is imposed. But, unbounded observables occur naturally in quantum mechanics and quantum probability theory [10, 11, 13, 23, 26] and so it is natural to consider the non-commutative integration in algebras of unbounded observables.

2 Pleriminaries

In this section we introduce the basic definition and properties of patial $*$-algebras [5] and partial O*-algebras [7].

A *partial $*$-algebra* is a complex vector space $\mathcal{A}$ with an involution $x \to x^*$ and a subset $\Gamma \subset \mathcal{A} \times \mathcal{A}$ such that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$, for all $\lambda, \mu \in \mathbb{C}$;

(iii) whenever $(x, y) \in \Gamma$, there exists a product $x \cdot y \in \mathcal{A}$ with the usual properties of the multiplication:

$x \cdot (y + \lambda z) = x \cdot y + \lambda(x \cdot z)$ and $(x \cdot y)^* = y^* \cdot x^*$ for $(x, y), (x, z) \in \Gamma$ and $\lambda \in \mathbb{C}$.
The element e of the $\mathcal{A}$ is called a *unit* if $e^* = e, (e, x) \in \Gamma$ for all $x \in \mathcal{A}$, and $e \cdot x = x \cdot e = x$, for all $x \in \mathcal{A}$. Notice that the partial multiplication is not required to be associative. Whenever $(x, y) \in \Gamma$, x is called a *left multiplier* of y and y is called a *right multiplier* of x, and we write $x \in L(y)$ and $y \in R(x)$. For a subset $\mathcal{B} \subset \mathcal{A}$, we write $L(\mathcal{B}) = \bigcap_{x \in \mathcal{B}} L(x), R(\mathcal{B}) = \bigcap_{x \in \mathcal{B}} R(x)$.

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot|\cdot)$ and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D}, \mathcal{H})$ the set of all closable linear operators X in $\mathcal{H}$ such that $\mathcal{D}(X) = \mathcal{D}$.

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Then $\mathcal{L}(\mathcal{D},\mathcal{H})$ is a vector space with the usual operations: $X+Y, \lambda X$. A subset (resp. subspace) of $\mathcal{L}(\mathcal{D},\mathcal{H})$ is called an *O-family* (resp. *O-vector space*) on $\mathcal{D}$. We put

$$\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) = \{X \in \mathcal{L}(\mathcal{D},\mathcal{H}); \mathcal{D}(X^*) \supset \mathcal{D}\},$$
$$\mathcal{L}^\dagger(\mathcal{D}) = \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}); X\mathcal{D} \subset \mathcal{D} \text{ and } X^*\mathcal{D} \subset \mathcal{D}\}.$$

Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra with the usual operations: $X+Y, \lambda X, XY$ and the involutions $X \mapsto X^\dagger \equiv X^* \lceil \mathcal{D}$, and a $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is called an *O^*-algebra* on $\mathcal{D}$. We equip $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ with the usual sum $X+Y$, the scalar multiplication λX, the involution $X \mapsto X^\dagger \equiv X^* \lceil \mathcal{D}$ and the weak partial multiplication $X \Box Y = X^{\dagger *}Y$, defined whenever X is a weak left multiplier of Y, ($X \in L(Y)$ or $Y \in R(X)$). A partial $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is called a *partial O^*-algebra* on $\mathcal{D}$.

Let $\mathcal{M}$ be a partial O*-algebra on $\mathcal{D}$. The locally convex topology on $\mathcal{D}$ defined by the family $\{\|\cdot\|_X; X \in \mathcal{M}\}$ of seminorms $\|\xi\|_X = \|\xi\| + \|X\xi\|, \xi \in \mathcal{D}$ is called the *graph topology* on $\mathcal{D}$ and denoted by $t_\mathcal{M}$. The completion of $\mathcal{D}[t_\mathcal{M}]$ is denoted by $\widetilde{\mathcal{D}}[t_\mathcal{M}]$. If the locally convex space $\mathcal{D}[t_\mathcal{M}]$ is complete, then $\mathcal{M}$ is called *closed*. We also define the following domains:

$$\widehat{\mathcal{D}}(\mathcal{M}) = \bigcap_{X \in \mathcal{M}} \mathcal{D}(\overline{X}), \quad \mathcal{D}^*(\mathcal{M}) = \bigcap_{X \in \mathcal{M}} \mathcal{D}(X^*), \mathcal{D}^{**}(\mathcal{M}) = \bigcap_{X \in \mathcal{M}} \mathcal{D}((X^* \lceil \mathcal{D}^*(\mathcal{M}))^*),$$

and then

$$\mathcal{D} \subset \widetilde{\mathcal{D}}(\mathcal{M}) \subset \widehat{\mathcal{D}}(\mathcal{M}) \subset \mathcal{D}^{**}(\mathcal{M}) \subset \mathcal{D}^*(\mathcal{M}).$$

The partial O*-algebra $\mathcal{M}$ is called *fully closed* if $\mathcal{D} = \widehat{\mathcal{D}}(\mathcal{M})$, *self-adjoint* if $\mathcal{D} = \mathcal{D}^*(\mathcal{M})$, *essentially self-adjoint* if $\mathcal{D}^*(\mathcal{M}) = \widehat{\mathcal{D}}(\mathcal{M})$ and *algebraically self-adjoint* if $\mathcal{D}^*(\mathcal{M}) = \mathcal{D}^{**}(\mathcal{M})$.

We define two weak commutants of $\mathcal{M}$. The *weak bounded commutant* $\mathcal{M}'_{\mathrm{w}}$ of $\mathcal{M}$ is the set

$$\mathcal{M}'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}); (CX\xi|\eta) = (C\xi|X^\dagger\eta) \text{ for every } X \in \mathcal{M} \text{ and } \xi, \eta \in \mathcal{D}\}.$$

But the partial multiplication is not required to be associative, so we define the *quasi-weak bounded commutant* $\mathcal{M}'_{\mathrm{qw}}$ of $\mathcal{M}$ is the set

$$\mathcal{M}'_{\mathrm{qw}} = \{C \in \mathcal{M}'_{\mathrm{w}}; (CX_1^\dagger\xi|X_2\eta) = (C\xi|(X_1 \Box X_2)\eta) \text{ for all } X_1 \in L(X_2) \text{ and } \xi, \eta \in \mathcal{D}\}.$$

In general, $\mathcal{M}'_{\mathrm{qw}} \subsetneq \mathcal{M}'_{\mathrm{w}}$.

We define the notion of strongly cyclic vector for a partial O*-algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$. A vector ξ_0 in $\mathcal{D}$ is said to be *strongly cyclic* if $R^{\mathrm{w}}(\mathcal{M})\xi_0$ is dense in $\mathcal{D}[t_\mathcal{M}]$, and ξ_0 is said to be *separating* if $\overline{\mathcal{M}'_{\mathrm{w}}\xi_0} = \mathcal{H}$, where $R^{\mathrm{w}}(\mathcal{M}) = \{Y \in \mathcal{M}; X \Box Y \text{ is well-defined }, {}^\forall X \in \mathcal{M}\}$.

We introduce the notions of partial GW*-algebras and partial EW*-algebras which are unbounded generalizations of von Neumann algebras. A fully closed partial O*-algebra $\mathcal{M}$ on $\mathcal{D}$ is called a *partial GW*-algebra* if there exists a von Neumann algebra $\mathcal{M}_0$ on $\mathcal{H}$ such that $\mathcal{M}_0'\mathcal{D} \subset \mathcal{D}$ and $\mathcal{M} = [\mathcal{M}_0 \lceil \mathcal{D}]^{s^*}$. A partial O*-algebra $\mathcal{M}$ on $\mathcal{D}$ is said to be a *partial EW*-algebra* if $\overline{\mathcal{M}_b} \equiv \{A \in \mathcal{B}(\mathcal{H}); A\lceil \mathcal{D} \in \mathcal{M}\}$ is a von Neumann algebra, $\mathcal{M}_b\mathcal{D} \subset \mathcal{D}$ and $\overline{\mathcal{M}_b}'\mathcal{D} \subset \mathcal{D}$.

A *$*$-representation* of a partial $*$-algebra $\mathcal{A}$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, satisfying $\pi(e) = I$ whenever $e \in \mathcal{A}$, that is,

(i) π is linear;

(ii) $x \in L^{\mathrm{w}}(y)$ in $\mathcal{A}$ implies $\pi(x) \in L^{\mathrm{w}}(\pi(y))$ and $\pi(x)\Box\pi(y) = \pi(xy)$;
(iii) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathcal{A}$.

Let π be a $*$-representation of a partial $*$-algebra $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. Then we define

$$\begin{cases} \widetilde{\mathcal{D}}(\pi): \text{ the completion of } \mathcal{D} \text{ with respect to the graph topology } t_{\pi(\mathcal{A})}, \\ \widetilde{\pi}(x) = \overline{\pi(x)}\lceil\widetilde{\mathcal{D}}(\pi), \quad x \in \mathcal{A}; \end{cases}$$

$$\begin{cases} \widehat{\mathcal{D}}(\pi) = \bigcap_{x\in\mathcal{A}} \mathcal{D}(\overline{\pi(x)}), \\ \widehat{\pi}(x) = \overline{\pi(x)}\lceil\widehat{\mathcal{D}}(\pi), \quad x \in \mathcal{A}; \end{cases}$$

$$\begin{cases} \mathcal{D}^*(\pi) = \bigcap_{x\in\mathcal{A}} \mathcal{D}(\pi(x)^*), \\ \pi^*(x) = \pi(x^*)^*\lceil\mathcal{D}^*(\pi), \quad x \in \mathcal{A}. \end{cases}$$

We say that π is *closed* if $\mathcal{D} = \widetilde{\mathcal{D}}(\pi)$; *fully closed* if $\mathcal{D} = \widehat{\mathcal{D}}(\pi)$; *essentially self-adjoint* if $\widehat{\mathcal{D}}(\pi) = \mathcal{D}^*(\pi)$; *self-adjoint* if $\mathcal{D} = \mathcal{D}^*(\pi)$.

We introduce the *weak* and the *quasi-weak* commutants of a $*$-representaion π of a partial $*$-algebra $\mathcal{A}$ as follows:

$$\pi(\mathcal{A})'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}); (C\xi|\pi(x)\eta) = (C\pi(x^*)\xi|\eta) \text{ for all } x \in \mathcal{A} \text{ and } \xi,\eta \in \mathcal{D}(\pi)\}$$

and

$$\mathcal{C}_{\mathrm{qw}}(\pi) = \{C \in \pi(\mathcal{A})'_{\mathrm{w}}; (C\pi(x_1^*)\xi|\pi(x_2)\eta) = (C\xi|\pi(x_1x_2)\eta) \\ \text{for all } x_1, x_2 \in \mathcal{A} \text{ such that } x_1 \in L(x_2) \text{ and all } \xi,\eta \in \mathcal{D}(\pi)\},$$

respectively.

3 Radon-Nikodým theorem for biweights and regular biweights on partial $*$-algebras

In this section, we investigate Radon-Nikodým theorem for biweights and regular biweights on partial $*$-algebras.

3.1 Radon-Nikodým Theorem

We first state the notion of biweights on partial $*$-algebras. For a biweight, it possible to construct the GNS-representation, which gives a representation, of partial $*$-algebras.

Let $\mathcal{A}$ be a partial $*$-algebra. A sesquilinear form φ on $D(\varphi) \times D(\varphi)$, where $D(\varphi)$ is a subspace of $\mathcal{A}$, is said to be *positive* if $\varphi(x,x) \geq 0, {}^\forall x \in D(\varphi)$. Let φ be a positive sesquilinear form on $D(\varphi) \times D(\varphi)$. Then we have

(3.1.1) $$\varphi(x,y) = \overline{\varphi(y,x)}, {}^\forall x,y \in D(\varphi),$$

(3.1.2) $$|\varphi(x,y)|^2 \leq \varphi(x,x)\varphi(y,y), \quad {}^\forall x,y \in D(\varphi).$$

We put

$$\mathcal{N}_\varphi = \{x \in D(\varphi); \varphi(x,x) = 0\}.$$

By (3.1.2) we have

$$\mathcal{N}_\varphi = \{x \in D(\varphi); \varphi(x,y) = 0 \text{ for all } y \in D(\varphi)\},$$

and so $\mathcal{N}_\varphi$ is a subspace of $D(\varphi)$. We put

$$D(\varphi)/\mathcal{N}_\varphi = \{\lambda_\varphi(x) \equiv x + \mathcal{N}_\varphi ; x \in D(\varphi)\}.$$

Then $D(\varphi)/\mathcal{N}_\varphi$ is a pre-Hilbert space with inner product $(\lambda_\varphi(x)|\lambda_\varphi(y)) = \varphi(x,y)$, $x, y \in D(\varphi)$. We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by the completion of $D(\varphi)/\mathcal{N}_\varphi$.

Let φ be a positive sesquilinear form on $D(\varphi) \times D(\varphi)$. A subspace $B(\varphi)$ of $D(\varphi)$ is said to be *core* for φ if it is satisfied the following

(i) $B(\varphi) \subset R(\mathcal{A})$;
(ii) $\{ax; a \in \mathcal{A}, x \in B(\varphi)\} \subset D(\varphi)$;
(iii) $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$;
(iv) $\varphi(ax, y) = \varphi(x, a^*y), {}^\forall a \in \mathcal{A}, {}^\forall x, y \in B(\varphi)$;
(v) $\varphi(a^*x, by) = \varphi(x, (ab)y), \quad {}^\forall a \in L(b), {}^\forall x, y \in B(\varphi)$.

And we denote by $\mathcal{B}_\varphi$ the set of all cores $B(\varphi)$ for φ. A positive sesquilinear form φ on $D(\varphi) \times D(\varphi)$ such that $\mathcal{B}_\varphi \neq \emptyset$ is called a *biweight* on $\mathcal{A}$.

Let φ be a biweight on $\mathcal{A}$ with core $B(\varphi)$. We put

$$\pi^\circ_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathcal{A}, x \in B(\varphi),$$

then π°_φ is a $*$-representation of $\mathcal{A}$. We denote by π^B_φ the closure of π°_φ. The triple $(\pi^B_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ is called the *GNS-representation* for the biweight φ on $\mathcal{A}$ with core $B(\varphi)$. Throughout the section, $\mathcal{A}$ denotes a partial $*$-algebra and φ is a biweight on $\mathcal{A}$ with a fixed core $B(\varphi)$.

Next we give the Radon-Nikodým theorem of biweights.

Definition 3.1.1. Let ψ_1 and ψ_2 be biweights on $\mathcal{A}$. We say ψ_2 is an *extension* of ψ_1 and write $\psi_1 \subset \psi_2$, if

(i) $D(\psi_1) \subset D(\psi_2)$;
(ii) $\psi_1 = \psi_2$ on $D(\psi_1) \times D(\psi_1)$;
(iii) there exist a core $B(\psi_1)$ for ψ_1 and a core $B(\psi_2)$ for ψ_2 such that $B(\psi_1) \subset B(\psi_2)$.

Definition 3.1.2. (1) A biweight ψ of $\mathcal{A}$ is said to be *φ-dominated* and denoted by $\psi \leq r\varphi$ if

(i) $D(\varphi) \subset D(\psi)$;
(ii) ${}^\exists r > 0, \psi(x,x) \leq r\varphi(x,x), \quad {}^\forall x \in D(\varphi)$;
(iii) there exists a core $B(\psi)$ for ψ such that $B(\varphi) \subset B(\psi)$.

(2) A biweight ψ of $\mathcal{A}$ is said to be *φ-absolutely continuous* if

(i) $D(\varphi) \subset D(\psi)$;
(ii) the map $\lambda_\varphi(x) \to \lambda_\psi(x), \quad x \in D(\varphi)$ is closable;
(iii) there exists a core $B(\psi)$ for ψ such that $B(\varphi) \subset B(\psi)$.

(3) A biweight ψ on $\mathcal{A}$ is said to be *φ-singular* if

(i) $D(\varphi) \subset D(\psi)$;
(ii) for any $x \in D(\varphi)$ there exists a sequence $\{x_n\}$ in $D(\varphi)$ such that $\lim_{n\to\infty} \lambda_\varphi(x_n) = 0$ and $\lim_{n\to\infty} \lambda_\psi(x_n) = \lambda_\psi(x)$;
(iii) there exists a core $B(\psi)$ for ψ such that $B(\varphi) \subset B(\psi)$.

For each (1), (2), (3), if a biweight ψ satisfies

(iii)$'$ $B(\varphi)$ is a core for ψ,

then ψ is said to be *uniformly φ-dominated, uniformly φ-absolutely continuous* and *uniformly φ-singular*, respectively.

Lemma 3.1.3. *Let H be a positive self-adjoint operator in $\mathcal{H}_\varphi$ affiliated with $\mathcal{C}_{\rm qw}(\pi_\varphi^B)''$ such that $\mathcal{D}(H) \supset \lambda_\varphi(D(\varphi))$ and $H\lambda_\varphi(B(\varphi))$ is dense in $H\lambda_\varphi(D(\varphi))$. We put*

$$\begin{cases} D(\varphi_{H,H}) = D(\varphi) \\ \varphi_{H,H}(x,y) = (H\lambda_\varphi(x)|H\lambda_\varphi(y)), \quad x,y \in D(\varphi). \end{cases}$$

Then $\varphi_{H,H}$ is a uniformly φ-absolutely continuous biweight on $\mathcal{A}$ with core $B(\varphi)$.

The following proposition is known as the Radon-Nikodým theorem for uniformly dominated biweights on $\mathcal{A}$.

Proposition 3.1.4. *Let ψ be a φ-dominated biweight ($\psi \le r\varphi$) on $\mathcal{A}$. Then there exists a positive operator K in $\mathcal{C}_{\rm qw}(\pi_\varphi^B)$ such that $0 \le K \le rI$ and $\varphi_K \subset \psi$, where*

$$\varphi_K(x,y) \equiv (K\lambda_\varphi(x)|\lambda_\varphi(y)), \quad x,y \in D(\varphi).$$

If $\mathcal{C}_{\rm qw}(\pi_\varphi^B)$ is a von Neumann algebra, then $H \equiv K^{\frac{1}{2}} \in \mathcal{C}_{\rm qw}(\pi_\varphi^B)$ and $\varphi_{H,H} \subset \psi$.

We shall show the Radon-Nikodým theorem for absolutely continuous biweights. For that, we prepare the following lemma.

Lemma 3.1.5. *Let φ_1 and φ_2 be biweights on $\mathcal{A}$ such that $\lambda_{\varphi_1+\varphi_2}(B(\varphi_1+\varphi_2))$ is dense in $\lambda_{\varphi_1+\varphi_2}(D(\varphi_1+\varphi_2))$. Then $\varphi_1+\varphi_2$ is a biweight on $\mathcal{A}$ and the following statements hold:*

(1) *If φ_1 and φ_2 are φ-absolutely continuous, then $\varphi_1+\varphi_2$ is an φ-absolutely continuous.*

(2) *If φ_1 is φ_2-dominated and φ_2 is (uniformly) φ-singular, then φ_1 is (uniformly) φ-singular.*

Theorem 3.1.6. *(Radon-Nikodým theorem) Let φ be a biweight on $\mathcal{A}$ with core $B(\varphi)$. Suppose that ψ is an φ-absolutely continuous biweight on $\mathcal{A}$ satisfies $\lambda_{\varphi+\psi}(B(\varphi))$ is dense in $\mathcal{H}_{\varphi+\psi}$. Then there exists a positive self-adjoint operator H in $\mathcal{H}_\varphi$ affiliated with $\mathcal{C}_{\rm qw}(\pi_\varphi^B)''$ such that*

(i) *$\lambda_\varphi(D(\varphi)) \subset \mathcal{D}(H)$;*

(ii) *$\varphi_{H,H}$ is a biweight on $\mathcal{A}$ with $D(\varphi_{H,H}) = D(\varphi)$ and $B(\varphi_{H,H}) = B(\varphi)$;*

(iii) *$\varphi_{H,H} \subset \psi$.*

Theorem 3.1.7. *(Lebesgue decomposition theorem) Let φ be a biweight on $\mathcal{A}$ with core $B(\varphi)$. Suppose ψ is a biweight on $\mathcal{A}$ such that*

(i) *$B(\varphi) \subset B(\psi)$;*

(ii) *$\lambda_{\varphi+\psi}(B(\varphi))$ is dense in $\mathcal{H}_{\varphi+\psi}$;*

(iii) *$\mathcal{C}_{\rm qw}(\pi_{\varphi+\psi}^B)$ is a von Neumann algebra.*

Then ψ is decomposed into $\psi = \psi_c + \psi_s$, where ψ_c is an φ-absolutely continuous biweight on $\mathcal{A}$ and ψ_s is a φ -singular biweight on $\mathcal{A}$.

3.2 Regular biweights on partial $*$-algebras

In this section we first give some biweight by a net of biweights on $\mathcal{A}$. Throughout this section let $\mathcal{A}$ be a partial $*$-algebra with identity e. Let $\{\varphi_\alpha\}$ be a net of biweights on $\mathcal{A}$. We put

$$
\begin{aligned}
D(\bigvee_\alpha \varphi_\alpha) &= \{x \in \bigcap_\alpha D(\varphi_\alpha); \sup_\alpha \varphi_\alpha(x,x) < \infty\},\\
(\bigvee_\alpha \varphi_\alpha)(x,y) &= \frac{1}{4}\{\sup_\alpha \varphi_\alpha(x+y,x+y) - \sup_\alpha \varphi_\alpha(x-y,x-y)\\
&\qquad + i\sup_\alpha \varphi_\alpha(x+iy,x+iy) - i\sup_\alpha \varphi_\alpha(x-iy,x-iy)\},\\
&\qquad\qquad {}^\forall x,y \in D(\bigvee_\alpha \varphi_\alpha).
\end{aligned}
$$

Then $D(\bigvee_\alpha \varphi_\alpha)$ is a subspace of $\mathcal{A}$, but $\bigvee_\alpha \varphi_\alpha$ is not a positive sesquilinear form in general. Therefore, we investigate conditions under which $\bigvee_\alpha \varphi_\alpha$ is a positive sesquilinear form.

Definition 3.2.1. A net $\{\varphi_\alpha\}$ of biweights on $\mathcal{A}$ is said that $\{\varphi_\alpha\}$ *has a net property* if for each finite subset $\{x_1, x_2, \cdots, x_m\}$ of $D(\bigvee_\alpha \varphi_\alpha)$ there exists a subsequence $\{\alpha_n\}$ in $\{\alpha\}$ such that $\lim_{n\to\infty} \varphi_{\alpha_n}(x_k,x_k) = \sup_\alpha \varphi_\alpha(x_k,x_k)$ for $k = 1, 2, \cdots, m$.

We have the following result:

Lemma 3.2.2. *Let $\{\varphi_\alpha\}$ be a net of biweights on $\mathcal{A}$. Then $\{\varphi_\alpha\}$ has a net property if and only if $\bigvee_\alpha \varphi_\alpha$ is a positive sesquiliear form on $D(\bigvee_\alpha \varphi_\alpha) \times D(\bigvee_\alpha \varphi_\alpha)$.*

It is easily shown that $\bigvee_\alpha \varphi_\alpha$ is a biweight under the following condition.

Lemma 3.2.3. *Let $\{\varphi_\alpha\}$ has a net property. Suppose $\bigcap_\alpha B(\varphi_\alpha)$ satisfies that*

$$
{}^\forall x \in D(\bigvee_\alpha \varphi_\alpha), {}^\exists\{x_n\} \subset \bigcap_\alpha B(\varphi_\alpha) \ \text{s.t.} \lim_{n\to\infty} (\sup_\alpha \varphi_\alpha)(x_n - x, x_n - x) = 0.
$$

Then, $\bigvee_\alpha \varphi_\alpha$ is a biweight on $\mathcal{A}$ with a core $\bigcap_\alpha B(\varphi_\alpha)$.

Next we define the notions of regularity and singularity of biweights, and give the decomposition theorem of biweights into the regular part and singular part.

Definition 3.2.4. A biweight φ is said to be *regular* if there exists a net $\{\varphi_\alpha\}$ of positive sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ such that $\varphi_\alpha \leq \varphi$ for all α and $\varphi(x,x) = \sup_\alpha \varphi_\alpha(x,x)$ for each $x \in D(\varphi)$, and it is said to be *singular* if there doesn't exist any positive sesquilinear form ψ on $\mathcal{A}$ such that $\psi \leq \varphi$ and $\psi \neq 0$.

We shall investigate the regularity of biweights. For that, we define trio-commutants $T(\varphi)'_{\rm w}$ and $T(\varphi)'_c$ for a biweight φ which play an important rule in regularity of φ as follows:

$$
\begin{aligned}
T(\varphi)'_{\rm w} &= \{K = (C,\xi,\eta); C \in \pi^B_\varphi(\mathcal{A})'_{\rm qw}, \xi,\eta \in \mathcal{D}^*(\pi^B_\varphi)\\
&\qquad \text{s.t. } C\lambda_\varphi(x) = (\pi^B_\varphi)^*(x)\xi, C^*\lambda_\varphi(x) = (\pi^B_\varphi)^*(x)\eta, {}^\forall x \in B(\varphi)\},\\
T(\varphi)'_c &= \{K = (C,\xi,\eta) \in T(\varphi)'_{\rm w}; \xi,\eta \in \mathcal{D}(\pi^B_\varphi)\}.
\end{aligned}
$$

For $K = (C, \xi, \eta) \in T(\varphi)'_{\mathrm{w}}$ we put

$$\pi'(K) = C, \quad \lambda'(K) = \xi, \quad \lambda'_*(K) = \eta.$$

Then we have the following

Lemma 3.2.5. (1) *$T(\varphi)'_{\mathrm{w}}$ is a $*$-invariant vector space under the following operators and the involution:*

$$K_1 + K_2 = (C_1 + C_2, \xi_1 + \xi_2, \eta_1 + \eta_2), \quad \alpha K = (\alpha C, \alpha\xi, \overline{\alpha}\eta), \quad K^* = (C^*, \eta, \xi)$$

for $K_1 = (C_1, \xi_1, \eta_1), K_2 = (C_2, \xi_2, \eta_2)$ and $K = (C, \xi, \eta)$ in $T(\varphi)'_{\mathrm{w}}$ and $\alpha \in \mathbb{C}$.
(2) $T(\varphi)'_c$ is a $*$-invariant subspace of $T(\varphi)'_{\mathrm{w}}$. In particular, if $\pi^B_\varphi(\mathcal{A})'_{\mathrm{w}}\mathcal{D}(\pi^B_\varphi) \subset D(\pi^B_\varphi)$, then $T(\varphi)'_c$ is a $*$-algebra under the following multiplication:

$$K_1K_2 = (C_1C_2, C_1\xi_2, C_2^*\eta_1)$$

for $K_1 = (C_1, \xi_1, \eta_1)$ and $K_2 = (C_2, \xi_2, \eta_2)$ in $T(\varphi)'_c$, and π' is a $*$-homomorphism of $T(\varphi)'_c$ into the von Neumann algebra $\pi^B_\varphi(\mathcal{A})'_{\mathrm{w}}$ and λ' is a linear map of $T(\varphi)'_c$ into $D(\pi^B_\varphi)$ satisfying $\pi'(K_1)\lambda'(K_2) = \lambda'(K_1K_2)$ for all $K_1, K_2 \in T(\varphi)'_c$.

Lemma 3.2.6. *Let φ be a biweight on $\mathcal{A}$ with a core $B(\varphi)$. Suppose ψ is a positive sesquilinear form on $\mathcal{A}$ such that $\psi \leq \varphi$. Then there exists an element $K \in T(\varphi)'_{\mathrm{w}}$ such that $0 \leq \pi'(K) \leq I$ and $\psi(x, e) = (\lambda_\varphi(x)|\lambda'(K))$ for all $x \in D(\varphi)$.*

For the regularity and the singularity of biweights we have following

Theorem 3.2.7. *Let φ be a biweight on $\mathcal{A}$ with a core $B(\varphi)$. Consider the following statements:*

(i) *There exists a net $\{K_\alpha\}$ in $T(\varphi)'_c$ such that $0 \leq \pi'(K_\alpha) \leq I$ for each α and $\pi'(K_\alpha) \to I$ strongly.*

(ii) *There exists a net $\{\xi_\alpha\}$ in $D(\pi^B_\varphi)$ such that $\varphi(x, x) = \sup_\alpha \|\pi^B_\varphi(x)\xi_\alpha\|^2$ for all $x \in D(\varphi)$.*

(iii) *φ is regular.*

(iv) *There exists a net $\{K_\alpha\}$ in $T(\varphi)'_{\mathrm{w}}$ such that $0 \leq \pi'(K_\alpha) \leq I$ for each α and $\pi'(K_\alpha) \to I$ strongly.*

Then the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold. In particular, the statements (i) $\sim$ (iv) are equivalent when π^B_φ is self-adjoint.

Theorem 3.2.8. *Suppose φ is a biweight on $\mathcal{A}$ with a core $B(\varphi)$ such that π^B_φ is self-adjoint. Then φ is singular if and only if there does not exist any element K of $T(\varphi)'_c$ such that $\pi'(K) \geq 0$ and $\pi'(K) \neq 0$.*

As the decomposition theorem of biweight we have following

Theorem 3.2.9. *Suppose φ is a biweight on $\mathcal{A}$ with a core $B(\varphi)$ such that π^B_φ is self-adjoint. Then φ is decomposed into $\varphi = \varphi_r + \varphi_s$, where φ_r is a regular biweight on $\mathcal{A}$ with $B(\varphi_r) = B(\varphi)$ and φ_s is a singular biweight on $\mathcal{A}$ with $B(\varphi_s) = B(\varphi)$ such that $\pi^B_{\varphi_r}, \pi^B_{\varphi_s}$ are self-adjoint.*

Notes of Section 3. Radon-Nikodým theorem for biweights and regular biweights on partial $*$-algebras are due to [20].
3.2. Theorem 3.2.7 and 3.2.9 are a generalization of Theorem 3.6 and 3.9 in [17] for weights on O*-algebra to biweights on partial $*$-algebras, respectively.

4 Trace representation of weights on partial O*-algebras

In this section, we investigate trace representation of weights on partial O*-algebras. This study is important for the structure of partial O*-algebras and for the applications to quantum physics. We extend arguments that are considerd for case of O*-algebras to case of partial O*-algebras.

4.1 Trace functionals

Throughout this section, suppose $\mathcal{H}$ is a *separable* Hilbert space. Let $\mathfrak{S}_1(\mathcal{H})$ be the set of all trace class operators on $\mathcal{H}$. Every operator T in $\mathfrak{S}_1(\mathcal{H})$ can be represented as $T = \sum_{n=1}^{\infty} t_n \xi_n \otimes \overline{\eta_n}$, where $\{t_n\} \subset \mathbb{C}$, $\sum_{n=1}^{\infty} |t_n| < \infty$ and $\{\xi_n\}_{n\in\mathbb{N}'}$ and $\{\eta_n\}_{n\in\mathbb{N}'}$ are orthonormal sets in $\mathcal{H}$ with $\mathbb{N}' = \{n \in \mathbb{N}; t_n \neq 0\}$. Furthermore, the trace norm $\nu(T) \equiv tr|T|$ equals $\sum_n |t_n|$. In case $T^* = T$ we can have in addition that $t_n \in \mathbb{R}$ and $\xi_n = \eta_n$ for all $n \in \mathbb{N}'$ [21]. Further, we put $\xi_n = \eta_n = 0$ for all $n \in \mathbb{N} \setminus \mathbb{N}'$. If the preceding conditions are fulfilled, then we call the sum $\sum_{n=1}^{\infty} t_n \xi_n \otimes \overline{\eta_n}$ a *canonical representation* of T. Let $\{\xi_n\}$ and $\{\eta_n\}$ be in $\mathcal{H}$. Suppose that $\sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \infty$. Then we can define a trace class operator $\sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n}$ by

$$\sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n} : x \in \mathcal{H} \to \sum_{n=1}^{\infty} (x|\eta_n)\xi_n \in \mathcal{H},$$

and then

$$tr(\sum_{n=1}^{\infty} \xi_n \otimes \overline{\eta_n}) = \sum_{n=1}^{\infty} (\xi_n|\eta_n) \tag{4.1.1}$$

([21], §42.5). Let $\mathfrak{M}$ be an O-family on $\mathcal{D}$ in $\mathcal{H}$. We define the following subsets of $\mathfrak{S}_1(\mathcal{H})$:

$$\begin{aligned}
\mathfrak{S}_1(\mathfrak{M}) &= \{T \in \mathfrak{S}_1(\mathcal{H}); T\mathcal{H} \subset \mathcal{D} \text{ and } XT \in \mathfrak{S}_1(\mathcal{H}), {}^{\forall}X \in \mathfrak{M}\},\\
\mathfrak{S}_{11}(\mathfrak{M}) &= \{T \in \mathfrak{S}_1(\mathcal{H}); T\mathcal{H} \subset \mathcal{D}, T^*\mathcal{H} \subset \mathcal{D} \text{ and } \overline{XTY^\dagger} \in \mathfrak{S}_1(\mathcal{H}), {}^{\forall}X, Y \in \mathfrak{M}\},\\
{}_1\mathfrak{S}(\mathfrak{M}) &= \{T \in \mathfrak{S}_1(\mathcal{H}); TX^\dagger \text{ is closable and } \overline{TX^\dagger} \in \mathfrak{S}_1(\mathcal{H}), {}^{\forall}X \in \mathfrak{M}\},\\
{}_{11}\mathfrak{S}(\mathfrak{M}) &= \{T \in \mathfrak{S}_1(\mathcal{H}); T\mathcal{H} \cup T^*\mathcal{H} \subset \bigcap_{X\in\mathfrak{M}} \mathcal{D}(X^{\dagger *})\\
&\qquad\qquad \text{and } \overline{X^{\dagger *}TY^\dagger} \in \mathfrak{S}_1(\mathcal{H}), {}^{\forall}X, Y \in \mathfrak{M}\}.
\end{aligned}$$

Their hermitian parts are denoted by $\mathfrak{S}_1(\mathfrak{M})_h, \mathfrak{S}_{11}(\mathfrak{M})_h, {}_1\mathfrak{S}(\mathfrak{M})_h$ and ${}_{11}\mathfrak{S}(\mathfrak{M})_h$, and their positive parts are denoted by $\mathfrak{S}_1(\mathfrak{M})_+$, $\mathfrak{S}_{11}(\mathfrak{M})_+$, ${}_1\mathfrak{S}(\mathfrak{M})_+$ and ${}_{11}\mathfrak{S}(\mathfrak{M})_+$. Then $\mathfrak{S}_1(\mathfrak{M}), \mathfrak{S}_{11}(\mathfrak{M}), {}_1\mathfrak{S}(\mathfrak{M})$ and ${}_{11}\mathfrak{S}(\mathfrak{M})$ are subalgebras of $\mathfrak{S}_1(\mathcal{H})$, their hermitian parts are real subspaces of $\mathfrak{S}_1(\mathcal{H})$ and their positive parts are positive cones in $\mathfrak{S}_1(\mathcal{H})$. Furthermore, they have the following properties:

Proposition 4.1.1. *Let $\mathfrak{M}$ be an O-family on $\mathcal{D}$ in $\mathcal{H}$. Then the following statements hold:*

(1)
$$\begin{array}{ccc}
\mathfrak{S}_{11}(\mathfrak{M})_\diamond & \subset & \mathfrak{S}_1(\mathfrak{M})_\diamond\\
\cap & & \cap\\
{}_{11}\mathfrak{S}(\mathfrak{M})_\diamond & \subset & {}_1\mathfrak{S}(\mathfrak{M})_\diamond,
\end{array}$$

where the symbol $\diamond$ means h or $+$.

(2) *We put*

$$\mathfrak{S}^1(\mathfrak{M})_\diamond = \{T = \sum_{n=1}^{\infty} t_n \xi_n \otimes \overline{\xi_n} \in \mathfrak{S}_1(\mathcal{H})_\diamond; T\mathcal{H} \subset \mathcal{D} \text{ and } \sum_{n=1}^{\infty} |t_n| \|X\xi_n\|^2 < \infty, {}^\forall X \in \mathfrak{M}\}$$

and

$${}^1\mathfrak{S}(\mathfrak{M})_\diamond = \{T = \sum_{n=1}^{\infty} t_n \xi_n \otimes \overline{\xi_n} \in \mathfrak{S}_1(\mathcal{H})_\diamond; T\mathcal{H} \subset \bigcap_{X \in \mathfrak{M}} \mathcal{D}(X^{\dagger *}) \text{ and } \sum_{n=1}^{\infty} |t_n| \|X^{\dagger *}\xi_n\|^2 < \infty, {}^\forall X \in \mathfrak{M}\},$$

where $T = \sum_{n=1}^{\infty} t_n \xi_n \otimes \overline{\xi_n}$ *is a canonical represenation of* T. *Then,*

$$\begin{array}{ccccc} \mathfrak{S}_{11}(\mathfrak{M})_+ = \mathfrak{S}^1(\mathfrak{M})_+ & \subset & \mathfrak{S}^1(\mathfrak{M})_h & \subset & \mathfrak{S}_{11}(\mathfrak{M})_h \\ \cap & & \cap & & \cap \\ {}_{11}\mathfrak{S}(\mathfrak{M})_+ = {}^1\mathfrak{S}(\mathfrak{M})_+ & \subset & {}^1\mathfrak{S}(\mathfrak{M})_h & \subset & {}_{11}\mathfrak{S}(\mathfrak{M})_h. \end{array}$$

(3) *Suppose that* $\mathfrak{M}$ *is an* O*-*family. Then the following statements are equivalent:*
(i) $\mathfrak{M}$ *is self-adjoint,*
(ii) $\mathfrak{S}_1(\mathfrak{M})_\diamond = {}_1\mathfrak{S}(\mathfrak{M})_\diamond$,
(iii) $\mathfrak{S}_{11}(\mathfrak{M})_\diamond = {}_{11}\mathfrak{S}(\mathfrak{M})_\diamond$.

(4) *Suppose that* $\mathfrak{M}$ *is a partial* O*-*algebra. We put*

$$\begin{aligned} \mathfrak{M}_{[2]} &= \{X \in \mathfrak{M}; X^\dagger \in L(X)\}, \\ \mathfrak{M}^{[2]} &= \text{ the linear span of } \{X^\dagger \Box X; X \in \mathfrak{M}_{[2]}\}. \end{aligned}$$

Then,

$$\begin{array}{ccccc} \mathfrak{S}_1(\mathfrak{M}^{[2]})_+ = \mathfrak{S}_{11}(\mathfrak{M}_{[2]})_+ & \subset & \mathfrak{S}_1(\mathfrak{M}^{[2]})_h & \subset & \mathfrak{S}_{11}(\mathfrak{M}_{[2]})_h \\ \cap & & \cap & & \cap \\ {}_1\mathfrak{S}(\mathfrak{M}^{[2]})_+ = {}_{11}\mathfrak{S}(\mathfrak{M}_{[2]})_+ & \subset & {}_1\mathfrak{S}(\mathfrak{M}^{[2]})_h & \subset & {}_{11}\mathfrak{S}(\mathfrak{M}_{[2]})_h. \end{array}$$

Every element T *of* $\mathfrak{S}_1(\mathfrak{M}^{[2]})^* \cap \mathfrak{S}_1(\mathfrak{M}^{[2]})$ *(resp.* ${}_1\mathfrak{S}(\mathfrak{M}^{[2]})^* \cap {}_1\mathfrak{S}(\mathfrak{M}^{[2]})$*) can be written as* $T = (T_1 - T_2) + i(T_3 - T_4)$ *with* $T_j \in \mathfrak{S}_1(\mathfrak{M}^{[2]})_+$ *(resp.* ${}_1\mathfrak{S}(\mathfrak{M}^{[2]})_+$*)* $(j = 1, \cdots, 4)$.

(5) *Suppose that* $\mathfrak{M}$ *is an* O*-*algebra. Then,*

$$\mathfrak{S}_1(\mathfrak{M}) = \mathfrak{S}_{11}(\mathfrak{M}) \text{ and } {}_1\mathfrak{S}(\mathfrak{M}) = {}_{11}\mathfrak{S}(\mathfrak{M}).$$

Every element T *of* $\mathfrak{S}_1(\mathfrak{M})^* \cap \mathfrak{S}_1(\mathfrak{M})$ *(resp.* ${}_1\mathfrak{S}(\mathfrak{M})^* \cap {}_1\mathfrak{S}(\mathfrak{M})$*) can be written as* $T = (T_1 - T_2) + i(T_3 - T_4)$ *with* $T_j \in \mathfrak{S}_1(\mathfrak{M})_+$ *(resp.* ${}_1\mathfrak{S}(\mathfrak{M})_+$*)* $(j = 1, \cdots, 4)$.

It is important for the study of unbounded Tomita-Takesaki theory [2, 3, 4, 11] to investigate the relation of $\mathfrak{S}_{11}(\mathfrak{M})_+$ and the space $\mathfrak{S}_2(\mathfrak{M})_+$ of positive Hilbert-Schmidt operators:

$$\begin{aligned} &\mathcal{H} \otimes \overline{\mathcal{H}}(\text{ or } \mathfrak{S}_2(\mathcal{H})) : \text{ the set of all Hilbert-Schmidt operators on } \mathcal{H}, \\ &\mathfrak{S}_2(\mathfrak{M})_+ = \{T \in \mathcal{B}(\mathcal{H})_+; T\mathcal{H} \subset \mathcal{D} \text{ and } XT \in \mathcal{H} \otimes \overline{\mathcal{H}} \text{ for each } X \in \mathfrak{M}\}. \end{aligned}$$

Then we have the following

Lemma 4.1.2. *Suppose that $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})_{[2]}$ is fully closed (it is so for example when $\mathcal{L}^\dagger(\mathcal{D})$ is closed) and $T \in \mathcal{B}(\mathcal{H})_+$ with $T\mathcal{H} \subset \mathcal{D}$. Then the following statements hold:*

(1) $T^\alpha \mathcal{H} \subset \mathcal{D}$ *for all* $\alpha > 0$.

(2) *Suppose that there exists an element N of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that $\overline{N}^{-1}\mathcal{H} \subset \mathcal{D}$ and $\overline{N}^{-1} \in \mathfrak{S}_1(\mathcal{H})$. Then, $T^\alpha\mathcal{H} \subset \mathcal{D}$ and $T^\alpha \in \mathfrak{S}_1(\mathcal{H})$ for each $\alpha > 0$.*

Proposition 4.1.3. *Suppose that $\mathfrak{M}$ is an O*-family on $\mathcal{D}$ in $\mathcal{H}$ and $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})_{[2]}$ is fully closed. Then,*

$$\mathfrak{S}_2(\mathfrak{M})_+ = \{T^{1/2}; T \in \mathfrak{S}_{11}(\mathfrak{M})_+\}.$$

Let $\mathfrak{M}$ be a partial O*-algebra on $\mathcal{D}$ in $\mathcal{H}$. We define two positive cones $\mathcal{P}(\mathfrak{M})$ and $\mathfrak{M}_+$ by

$$\mathcal{P}(\mathfrak{M}) = \{\sum_{k=1}^{n} X_k^\dagger \Box X_k; X_k \in \mathfrak{M}_{[2]}(k=1,\cdots,n), n \in \mathbb{N}\},$$
$$\mathfrak{M}_+ = \{X \in \mathfrak{M}; X \geq 0 \text{ iff } (X\xi|\xi) \geq 0, {}^\forall \xi \in \mathcal{D}\}.$$

A linear functional f on $\mathfrak{M}$ is said to be *positive* (resp. *strongly positive*) if $f(X) \geq 0$ for each $X \in \mathcal{P}(\mathfrak{M})$ (resp. $\mathfrak{M}_+$).
It is clear that every strongly positive linear functional on $\mathfrak{M}$ is positive, but the converse does not hold in general. Woronowicz [30] has given the following example: As it is well known the Schwartz space $\mathcal{S}(\mathbb{R})$ of the infinitely differentiable rapidly decreasing functions is dense in $L^2(\mathbb{R})$. Define the operators $Q : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $P : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by $Qf(x) = xf(x)$ and $Pf(x) = -i\frac{d}{dx}f(x)$. In physical literature Q and P are called the position and momentum operator respectively. Let $\mathfrak{M}_S$ be the O*-algebra on $\mathcal{S}(\mathbb{R})$ generated by Q and P. Set $A = (Q+iP)/\sqrt{2}$. The operator $N = AA^\dagger$ has $\mathbb{N} \cup \{0\}$ as its nondegenerate spectrum and for this reason it is called by physicists the number operator. It can be checked that $(A^\dagger A - I)(A^\dagger A - 2I) \notin \mathcal{P}(\mathfrak{M}_S)$. Hence there exists a positive linear functional f on $\mathfrak{M}_S$ such that $f((A^\dagger A - I)(A^\dagger A - 2I)) < 0$. Let $\{\xi_n\}$ be the orthonormal basis in $L^2(\mathbb{R})$ consisting of eigenvectors of the number operator $N \equiv AA^\dagger$. Since

$$((A^\dagger A - I)(A^\dagger A - 2I)\xi|\xi) = \sum_{n=0}^{\infty}(n-1)(n-2)|(\xi|\xi_n)|^2 \geq 0$$

for each $\xi \in \mathcal{S}(\mathbb{R})$, we have $(A^\dagger A - I)(A^\dagger A - 2I) \in \mathfrak{M}_+$, which implies that f is not strongly positive.

For any $T \in {}_1\mathfrak{S}(\mathfrak{M})$ we define linear functionals f_T and ${}_Tf$ on $\mathfrak{M}$ by

$$\begin{cases} f_T(X) = trX^{\dagger *}T, \\ {}_Tf(X) = tr\overline{TX}, \quad X \in \mathfrak{M}. \end{cases}$$

Then, for any $T \in {}_1\mathfrak{S}(\mathfrak{M})_h$,

$$f_T(X) = \sum_{n=1}^{\infty} t_n (X^{\dagger *}\xi_n|\xi_n), \quad {}_Tf(X) = \sum_{n=1}^{\infty} t_n(\xi_n|X^*\xi_n), \quad X \in \mathfrak{M},$$

where $T = \sum_{n=1}^{\infty} t_n \xi_n \otimes \overline{\xi_n}$ is a canonical representation of T, and so we have the following

Proposition 4.1.4. *Let $\mathfrak{M}$ be a partial O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Then the following statements hold:*

(1) *For any $T \in \mathfrak{S}_1(\mathfrak{M})_h$,*

$$f_T(X) = {}_Tf(X) = \sum_{n=1}^{\infty} t_n(X\xi_n|\xi_n), \quad X \in \mathfrak{M}.$$

If $T \in \mathfrak{S}_1(\mathfrak{M})_+$, then f_T is strongly positive.

(2) *Let $T \in {}_1\mathfrak{S}(\mathfrak{M})_h$. Then, $f_T = {}_Tf$ if and only if $\mathfrak{M}$ is algebraically self-adjoint.*

(3) *Every f_T, $T \in \mathfrak{S}_1(\mathfrak{M}^{[2]})^* \cap \mathfrak{S}_1(\mathfrak{M}^{[2]})$, is written as $f_T = (f_{T_1} - f_{T_2}) + i(f_{T_3} - f_{T_4})$ whereby $T_j \in \mathfrak{S}_1(\mathfrak{M}^{[2]})_+$ $(j = 1, \cdots, 4)$. In particular, if $\mathfrak{M}$ is an O^*-algebra on $\mathcal{D}$, then every f_T, $T \in \mathfrak{S}_1(\mathfrak{M})^* \cap \mathfrak{S}_1(\mathfrak{M})$, is written as $f_T = (f_{T_1} - f_{T_2}) + i(f_{T_3} - f_{T_4})$ whereby $T_j \in \mathfrak{S}_1(\mathfrak{M})_+$ $(j = 1, \cdots, 4)$.*

We remark that for $T \in {}_1\mathfrak{S}(\mathfrak{M})_+$ even if $f_T = {}_Tf$, f_T is not necessarily strongly positive.

4.2 Trace representation of weights

In this section we consider trace representation of weights on a partial O*-algebra which contains the inverse of a compact operator. Let $\mathfrak{M}$ be a fully closed partial O*-algebra on $\mathcal{D}$ in $\mathcal{H}$. A map φ of $\mathfrak{M}_+$ into $\mathbb{R}_+ \cup \{+\infty\}$ is said to be a *weight* on $\mathfrak{M}_+$ if

(W)$_1$ $\varphi(\alpha X) = \alpha\varphi(X), \alpha \geq 0, X \in \mathfrak{M}_+$;

(W)$_2$ $\varphi(X + Y) = \varphi(X) + \varphi(Y), \quad X, Y \in \mathfrak{M}_+$,

where $0 \cdot (+\infty) = 0$. Let φ be a weight on $\mathfrak{M}_+$. Refer to [4, 13, 14, 16, 17] for weights of O*-algebra and to [1, 2, 6, 9] for partial O*-algebras. We put

$$\mathfrak{N}_\varphi^0 = \{X \in \mathfrak{M}; X^\dagger \in L(X) \text{ and } \varphi(X^\dagger \square X) < \infty\}.$$

To consider trace representation of weights on partial O*-algebras, we will use an ordered $*$-vector space $L(\mathcal{D}, \mathcal{D}^\dagger)$ defined as follows. Given a dense linear subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$, we denote by $\mathcal{D}^\dagger$ the algebraic conjugate dual of $\mathcal{D}$, that is, the set of all conjugate linear functionals on $\mathcal{D}$. The set $\mathcal{D}^\dagger$ is a vector space under the following operations:

$$< v_1 + v_2, \xi > = < v_1, \xi > + < v_2, \xi >, < \alpha v, \xi > = \alpha < v, \xi >, \quad \xi \in \mathcal{D},$$

where $< v, \xi >$ is the value of $v \in \mathcal{D}^\dagger$ at $\xi \in \mathcal{D}$. We denote by $L(\mathcal{D}, \mathcal{D}^\dagger)$ the set of all linear maps from $\mathcal{D}$ to $\mathcal{D}^\dagger$. Then $L(\mathcal{D}, \mathcal{D}^\dagger)$ is a $*$-vector space under the usual operations: $S + T, \lambda T$ and the involution $T \to T^\dagger$ $(< T^\dagger\xi, \eta > = \overline{< T\eta, \xi >}, \xi, \eta \in \mathcal{D})$. Furthermore, $L(\mathcal{D}, \mathcal{D}^\dagger)_h \equiv \{T \in L(\mathcal{D}, \mathcal{D}^\dagger); T^\dagger = T\}$ is an ordered set under the order $S \leq T$ $(< S\xi, \xi > \leq < T\xi, \xi >, {}^\forall\xi \in \mathcal{D})$. We remark that any linear operator X defined on $\mathcal{D}$ is regarded as an element of $L(\mathcal{D}, \mathcal{D}^\dagger)$ by $< X\xi, \eta > = (X\xi|\eta)$, $\xi, \eta \in \mathcal{D}$. In particular, $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ are regarded as ordered $*$-subspaces of $L(\mathcal{D}, \mathcal{D}^\dagger)$. For any pair $X, Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$ we define the product $Y^\dagger \circ A \circ X$ by

$$< (Y^\dagger \circ A \circ X)\xi, \eta > = (AX\xi|Y\eta), \quad \xi, \eta \in \mathcal{D}.$$

Then we have

$$Y^\dagger \circ A \circ X \in L(\mathcal{D}, \mathcal{D}^\dagger)$$

and if A is hermitian and $X^\dagger \in L(X)$, then

$$X^\dagger \circ A \circ X \leq \|A\| X^\dagger \square X. \tag{4.2.1}$$

A pair $(\mathfrak{L},\mathfrak{N})$ of an O*-vector subspace $\mathfrak{L}$ of $\mathfrak{M}$ and a subset $\mathfrak{N}$ with I of $\mathfrak{M}$ is said to *satisfy the order condition* if:

$$X^\dagger \in L(X) \text{ and } X^\dagger \Box X \in \mathfrak{L}, {}^\forall X \in \mathfrak{N}.$$

We denote by $L_{(\mathfrak{L},\mathfrak{N})}$ the ordered $*$-vector space generated by $\mathfrak{L}$ and $\{X^\dagger \circ A \circ X; X \in \mathfrak{N}, A \in \mathcal{B}(\mathcal{H})\}$. Then,

$$\{X^\dagger \circ A, A \circ X; A \in \mathcal{B}(\mathcal{H}), X \in \mathfrak{N}\} \subset L_{(\mathfrak{L},\mathfrak{N})} \subset L(\mathcal{D},\mathcal{D}^\dagger).$$

Lemma 4.2.1. *Let $(\mathfrak{L},\mathfrak{N})$ be a pair satisfying the order condition. Then, every strongly positive linear functional f on $\mathfrak{L}$ can be extended to a positive linear functional $\widetilde{f}$ on $L_{(\mathfrak{L},\mathfrak{N})}$ such that*

$$\begin{aligned}
&\widetilde{f}(A \circ X) = tr A\overline{X}T,\\
&\widetilde{f}(X^\dagger \circ A) = tr A\overline{TX^\dagger},\\
&\widetilde{f}(X^\dagger \circ A \circ X) = tr A\overline{\overline{X}TX^\dagger}, \quad A \in \mathcal{F}(\mathcal{H}), X \in \mathfrak{N}
\end{aligned}$$

for some $T \in {}_{11}\mathfrak{S}(\mathfrak{N})_+$ s.t. $T\mathcal{H} \subset \bigcap_{X\in\mathfrak{N}} \mathcal{D}(\overline{X})$, where $\mathcal{F}(\mathcal{H})$ denotes the set of all bounded finite rank operators on $\mathcal{H}$.

Using Lemma 4.2.1 we can prove the following main theorem

Theorem 4.2.2. *Let $\mathfrak{M}$ be a partial O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ and let φ be a weight on $\mathfrak{M}_+$. Suppose that there exists an element N of $\mathfrak{N}_\varphi^0$ which has a positive self-adjoint extension $\widetilde{N}$ such that $\widetilde{N}^{-1}$ is a compact operator on $\mathcal{H}$. Then there exists an element T of ${}_{11}\mathfrak{S}(\mathfrak{N}_\varphi^0)_+$ such that $T\mathcal{H} \subset \bigcap_{X\in\mathfrak{N}_\varphi^0} \mathcal{D}(\overline{X})$ and*

(i) $\varphi(X^\dagger \Box X) = tr(T^{1/2}X^\dagger)^*\overline{T^{1/2}X^\dagger} = tr\overline{\overline{X}TX^\dagger}$ *for all $X \in \mathfrak{M}$ such that $X^\dagger \in L(X)$ and $N\Box X \in \mathfrak{N}_\varphi^0$ (this implies automatically $X \in \mathfrak{N}_\varphi^0$);*

(ii) $\varphi(X) = tr\overline{TX}$ *for each positive operator X in $\mathfrak{N}_\varphi^0$.*

Remark 4.2.3. *In statement (i) of Theorem 4.2.2 the condition $N\Box X \in \mathfrak{N}_\varphi^0$ may not be replaced by the weaker condition that $X \in \mathfrak{N}_\varphi^0$. We give an example: Let $\mathfrak{M}_S$ be the O^*-algebra on the Schwartz space $\mathcal{S}(\mathbb{R})$ generated by the momentum operator P and the position operator Q, and $N = \sum_{n=0}^\infty (n+1)\xi_n \otimes \overline{\xi_n}$ the number operator, where $\{\xi_n\}_{n=0,1,\cdots} \subset \mathcal{S}(\mathbb{R})$ is an ONB in $L^2(\mathbb{R})$ consisting of the Hermite functions [23, 30]. Let a weight φ on $(\mathfrak{M}_S)_+$ be defined by*

$$\varphi(X) = \lim_{n\to\infty} \frac{1}{n^2}(X\xi_n|\xi_n), \qquad X \in (\mathfrak{M}_S)_+.$$

Then $\varphi(I) = 0$ and $\varphi(N^2) = 1$. Furthermore, the pair $(\mathfrak{M}_S,\varphi)$ satisfies the assumptions of Theorem 4.2.2. Suppose now that there exists a positive trace class operator T on $\mathcal{H}$ such that $\varphi(X^\dagger X) = tr(T^{1/2}X^\dagger)^*\overline{T^{1/2}X^\dagger}$ for every $X \in \mathfrak{N}_\varphi^0$. Since $\varphi(I) = 0$, it follows that $T = 0$, which implies that $1 = \varphi(N^2) = tr(T^{1/2}N)^*\overline{T^{1/2}N} = 0$. This is a contradiction.

On the other hand, Theorem 4.2.2, (i) has the following natural generalization. Put $\mathcal{N}$ for the family of all $N \in \mathfrak{N}_\varphi^0$ such that N has positive self-adjoint extension $\widetilde{N}$ and $\widetilde{N}^{-1}$ is a compact operator on $\mathcal{H}$. Given $N \in \mathcal{N}$, set $\mathfrak{N}_\varphi^0(N) = \{X \in \mathfrak{M}; X^\dagger \in L(X)$ and $N\Box X \in \mathfrak{N}_\varphi^0\}$. Then we have

Corollary 4.2.4. *If $\mathfrak{M}$ is a partial O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ and φ a weight on $\mathfrak{M}_+$ such that $\mathcal{N}$ is not empty. Then there is $T \in {}_{11}\mathfrak{S}(\mathfrak{N}_\varphi^0)$ such that*

$$\mathcal{H} \subset \bigcap_{X \in \mathfrak{N}_\varphi^0} \mathcal{D}(\overline{X})$$

and

$$\varphi(X^\dagger \Box X) = tr(T^{1/2}X^\dagger)^* \overline{T^{1/2}X^\dagger} = tr\overline{\overline{X}TX^\dagger}$$

for all $X \in \bigcup_{N \in \mathcal{N}} \mathfrak{N}_\varphi^0(N)$.

We have the following

Corollary 4.2.5. *Let $\mathfrak{M}$ be a partial O*-algebra on $\mathcal{D}$. Suppose that there exists an element N of $\mathfrak{M}_{[2]}$ which has a positive self-adjoint extension $\widetilde{N}$ such that $\widetilde{N}^{-1}$ is a compact operator on $\mathcal{H}$. Then, for any strongly positive linear functional f on $\mathfrak{M}$ there exists an element T of ${}_{11}\mathfrak{S}(\mathfrak{M}_{[2]})_+$ such that $T\mathcal{H} \subset \bigcap_{X \in \mathfrak{M}_{[2]}} \mathcal{D}(\overline{X})$ and*

(i) *$f(X^\dagger \Box X) = tr(T^{1/2}X^\dagger)^* \overline{(T^{1/2}X^\dagger)} = tr\overline{\overline{X}TX}$ for all $X \in \mathfrak{M}$ such that $X \in \mathfrak{M}_{[2]}$ and $N \Box X \in \mathfrak{M}_{[2]}$;*

(ii) *$f(X) = tr\overline{TX}$ for each $X \in \mathfrak{M}_{[2]}$.*

In particular, if $\mathfrak{M}_{[2]}$ is fully closed then T can be taken in $\mathfrak{S}_1(\mathfrak{M}_{[2]})_+$. Furthermore, if $\mathfrak{M}$ is an O-algebra, then the above results hold for $\mathfrak{M}$ instead of $\mathfrak{M}_{[2]}$. In particular, if $\mathfrak{M}$ is a closed O*-algebra, then every strongly positive linear functional f on $\mathfrak{M}$ is of the form $f = f_T$ for some $T \in \mathfrak{S}_1(\mathfrak{M})_+$.*

Example 4.2.6. *Let P and Q be the momentum operator and the position operator on $\mathcal{S}(\mathbb{R})$, respectively. Let $\mathfrak{M}$ be a partial O*-algebra on $\mathcal{S}(\mathbb{R})$ containing P and Q. Let N be the number operator and $\mathfrak{M}_S$ the O*-algebra on $\mathcal{S}(\mathbb{R})$ generated by P and Q. Then, $N \in \mathfrak{M}_S$, $\overline{N} \geq I$ and $\overline{N}^{-1}$ is a compact operator. Furthermore, since $\mathfrak{M}_S \subset \mathfrak{M}_{[2]}^N \equiv \{X \in \mathfrak{M}; X, N \Box X \in \mathfrak{M}_{[2]}\} \subset \mathfrak{M}_{[2]} \subset \mathfrak{M}$, and $\mathfrak{M}_S$ is a self-adjoint O*-algebra on $\mathcal{S}(\mathbb{R})$, it follows from Theorem 4.2.2 that for any weight φ on $\mathfrak{M}_+$ satisfying $\varphi(N^2) < \infty$ there exists an element T of $\mathfrak{S}_1(\mathfrak{N}_\varphi^0)_+$ such that*

(i) *$\varphi(X^\dagger \Box X) = tr(T^{1/2}X^\dagger)^* \overline{T^{1/2}X^\dagger} = trX^{\dagger *}\overline{TX}$ for each $X \in \mathfrak{M}_{[2]}^N$;*

(ii) *$\varphi(X) = tr\overline{TX}$ for each $X \in (\mathfrak{N}_\varphi^0)_+$.*

Furthermore, it follows from Corollary 4.2.5 that for any strongly positive linear functional f on $\mathfrak{M}$ there exists an element T of $\mathfrak{S}_1(\mathfrak{M}_{[2]})_+$ such that $f(X) = trXT$ for all $X \in \mathfrak{M}_{[2]}$. In particular, every strongly positive linear functional f on $\mathfrak{M}_S$ is of the form $f = f_T$ for some $T \in \mathfrak{S}_1(\mathfrak{M}_S)_+$.

4.3 Trace representations of uniformly continuous linear functionals

We define the locally convex topology τ_u (resp. τ_c) on $\mathfrak{M}$ called *uniform topology* (resp. *precompact uniform topology*) determined by the family of seminorms:

$$p_{\mathcal{M},\mathcal{N}}(X) = \sup\{|(X\xi|\eta)|; \xi \in \mathcal{M}, \eta \in \mathcal{N}\}, \quad X \in \mathfrak{M},$$

where $\mathcal{M}$ and $\mathcal{N}$ range over all bounded subsets (resp. precompact subsets) of $\mathcal{D}[t_{\mathfrak{M}}]$. We have the following

Theorem 4.3.1. *Let $\mathfrak{M}$ be a fully closed partial O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Then, every continuous linear functional f on $\mathfrak{M}[\tau_c]$ is of the form*

$$f(X) = trXT, \quad X \in \mathfrak{M}$$

for some $T \in \mathfrak{S}_{11}(\mathfrak{M})$.

It is natural to consider the following question: If a continuous linear functional on $\mathfrak{M}[\tau_c]$ is strongly positive, then is the trace class operator T in Theorem 4.3.1 positive?

Proposition 4.3.2. (1) *Let $\mathfrak{M}$ be a fully closed partial O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Suppose that there exists an orthonormal basis $\{\xi_n\}$ in $\mathcal{H}$ such that $\{\xi_n \otimes \overline{\xi_n}; n \in \mathbb{N}\} \subset \mathfrak{M}$ (this holds in particular when $\mathfrak{M} = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$). Then, every strongly positive continuous linear functional f on $\mathfrak{M}[\tau_c]$ is of the form $f = f_T$ for some $T \in \mathfrak{S}_{11}(\mathfrak{M})_+$.*

(2) *Suppose that $\mathfrak{M}$ is an O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{D}[t_{\mathfrak{M}}]$ is a Fréchet Montel space. Then, every strongly positive linear functional f on $\mathfrak{M}$ is τ_u-continuous and it is of the form $f = f_T$ for some $T \in \mathfrak{S}_1(\mathfrak{M})_+$.*

4.4 Trace representation of regular weights

Let $\mathfrak{M}$ be a fully closed partial O*-algebra on $\mathcal{D}$ in $\mathcal{H}$. A weight φ on $\mathfrak{M}_+$ is said to be *regular* if $\varphi = \sup_\alpha f_\alpha$ for some net $\{f_\alpha\}$ of strongly positive linear functionals on $\mathfrak{M}$. We denote by $\mathcal{T}(\mathfrak{M})$ the collection of all nets $\{T_\alpha\}$ in $\mathfrak{S}_1(\mathfrak{M})_+$ such that the formula

$$\varphi_{\{T_\alpha\}}(X) = \sup_\alpha trXT_\alpha, \quad X \in \mathfrak{M}_+$$

defines a regular weight on $\mathfrak{M}_+$. We remark that $\varphi_{\{T_\alpha\}}$ ($\{T_\alpha\} \subset \mathfrak{S}_1(\mathfrak{M})_+$) is not necessarily a weight on $\mathfrak{M}_+$. The collections of all increasing and of all increasing and mutually commuting nets in $\mathfrak{S}_1(\mathfrak{M})_+$ are denoted by $\mathcal{T}_i(\mathfrak{M})$ and $\mathcal{T}_i^c(\mathfrak{M})$, respectively. It is clear that $\mathcal{T}_i^c(\mathfrak{M}) \subset \mathcal{T}_i(\mathfrak{M}) \subset \mathcal{T}(\mathfrak{M})$.

We can show the following result for trace representation of a weight $\varphi_{\{T_\alpha\}}$ defined by $\{T_\alpha\} \in \mathcal{T}_i(\mathfrak{M})$ or $\{T_\alpha\} \in \mathcal{T}(\mathfrak{M})$:

Theorem 4.4.1. *Let $\{T_\alpha\} \in \mathcal{T}(\mathfrak{M})$ be given such that $(\mathfrak{N}^0_{\varphi_{\{T_\alpha\}}})^\dagger \mathcal{D}$ is total in $\mathcal{H}$. Suppose that $\{T_\alpha\} \in \mathcal{T}_i(\mathfrak{M})$ or that $\mathcal{F}(\mathcal{H}) \subset \mathfrak{M}$. Then there exists a positive self-adjoint operator Ω in $\mathcal{H}$ such that*

$$\mathfrak{N}^0_{\varphi_{\{T_\alpha\}}} = \{X \in \mathfrak{M}_{[2]};\ \overline{\Omega X^\dagger} \in \mathcal{H} \otimes \overline{\mathcal{H}}\} \subset \{X \in \mathfrak{M}_{[2]};\ \overline{X^{\dagger *}\Omega} \in \mathcal{H} \otimes \overline{\mathcal{H}}\}$$

and

$$\varphi_{\{T_\alpha\}}(X^\dagger \square X) = tr(\Omega X^\dagger)^* \overline{\Omega X^\dagger} = tr(X^{\dagger *}\Omega)^* \overline{X^{\dagger *}\Omega}, \quad X \in \mathfrak{N}^0_{\varphi_{\{T_\alpha\}}}.$$

A partial O*-algebra $\mathfrak{M}$ is said to be *QMP-solvable* if every strongly positive linear functional f on $\mathfrak{M}$ is represented as

$$f(X) = trXT, \quad X \in \mathfrak{M}$$

for some $T \in \mathfrak{S}_1(\mathfrak{M})_+$. We shall consider trace representation of weights on QMP-solvable partial O*-algebras.

A weight φ on $\mathfrak{M}_+$ is said to be *m-regular* (or monotonously regular) if $\varphi = \sup_\alpha f_\alpha$ for some increasing net $\{f_\alpha\}$ of strongly positive linear functionals on $\mathfrak{M}$. It is said to be *sequentially m-regular* if $\varphi = \sup_n f_n$ for some increasing sequence $\{f_n\}$ of strongly positive linear functionals on $\mathfrak{M}$.

Suppose that φ is sequentially m-regular. Then it may be represented also as $\varphi = \sum_n g_n$, where $g_1 = f_1$ and $g_{n+1} = f_{n+1} - f_n$ are strongly positive as well. Since $\mathfrak{M}$ is QMP-solvable, φ is of the form $\varphi(X) = \sum_n tr\overline{T_n X}$ for some sequence $\{T_n\}$ in $\mathfrak{S}_1(\mathfrak{M})_+$. Thus Theorem 4.4.1 implies the following results.

Theorem 4.4.2. *Suppose that $\mathfrak{M}$ is a QMP-solvable partial O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ and that φ is a regular weight on $\mathfrak{M}$ such that $(\mathfrak{N}_\varphi^0)^\dagger\mathcal{D}$ is total in $\mathcal{H}$. Suppose further that φ is sequentially m-regular or that $\mathcal{F}(\mathcal{H}) \subset \mathfrak{M}$. Then there exists a positive self-adjoint operator Ω in $\mathcal{H}$ such that*

$$\mathfrak{N}_\varphi^0 = \{X \in \mathfrak{M}_{[2]};\ \overline{\Omega X^\dagger} \in \mathcal{H} \otimes \overline{\mathcal{H}}\} \subset \{X \in \mathfrak{M}_{[2]};\ \overline{X^{\dagger *}\Omega} \in \mathcal{H} \otimes \overline{\mathcal{H}}\},$$

$$\varphi(X^\dagger \square X) = tr(\Omega X^\dagger)^* \overline{\Omega X^\dagger} = tr(X^{\dagger *}\Omega)^* \overline{X^{\dagger *}\Omega}, \quad X \in \mathfrak{N}_\varphi^0.$$

Corollary 4.4.3. *Suppose that $\mathfrak{M}$ is a QMP-solvable partial O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ and that φ is a regular weight on $\mathfrak{M}_+$ satisfying $\varphi(I) < \infty$. Suppose further that φ is sequentially m-regular or that $\mathcal{F}(\mathcal{H}) \subset \mathfrak{M}$. Then there exists a positive Hilbert-Schmidt operator Ω on $\mathcal{H}$ such that*

$$\mathfrak{N}_\varphi^0 = \{X \in \mathfrak{M}_{[2]};\ \overline{\Omega X^\dagger} \in \mathcal{H} \otimes \overline{\mathcal{H}}\} = \{X \in \mathfrak{M}_{[2]};\ \overline{X^{\dagger *}\Omega} \in \mathcal{H} \otimes \overline{\mathcal{H}}\},$$
$$\varphi(X^\dagger \square X) = tr(\Omega X^\dagger)^* \overline{\Omega X^\dagger} = tr(X^{\dagger *}\Omega)^* \overline{X^{\dagger *}\Omega}, \quad X \in \mathfrak{N}_\varphi^0.$$

Notes of Section 4. Trace representations of weights on partial O*-algebras are due to [15].
4.2. The proof of Theorem 4.2.2 is shown similarly to that of Theorem 4.1 of [16]. Note that although our proof of part (ii) of Theorem 4.2.2 depends crucially on existence of an operator N satisfying the required assumptions the operator N itself does not enter into the final formulation. This is not so in part (i). The result in Corollary 4.2.5 for closed O*-algebras is a generalization of the Schmüdgen result [25, Theorem 2.2] for self-adjoint O*-algebras. The results stated in 4.3 and 4.4 are generalizations of those obtained for O*-algebras to partial O*-algebras, and they are proved almost the same way as corresponding statements concerning O*-algebra [16, 26].
4.3. The proof of Theorem 4.3.1 is shown similarly to that of Proposition 5.3.4 of [26].
4.4. The proof of Theorem 4.4.1 is shown similarly to that of Theorem 3.4 of [16].

5 Unbounded conditional expectations for partial O*-algebras

In this section, we consider unbounded conditional expectations for partial O*-algebras.

5.1 Weak conditional expectations

In this section let $\mathcal{M}$ be a self-adjoint partial O*-algebra containing the identity I on $\mathcal{D}$ in $\mathcal{H}$ with a strongly cyclic vector ξ_0 and let $\mathcal{N}$ be a partial O*-subalgebra of $\mathcal{M}$ such that

(N) $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}} \equiv \overline{\mathcal{N}\xi_0}$.

The following is easily shown.

Lemma 5.1.1. *We put*

$$\begin{cases} \mathcal{D}(\pi_{\mathcal{N}}) = (\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0 \\ \pi_{\mathcal{N}}(X)Y\xi_0 = (X\square Y)\xi_0, \quad {}^\forall X \in \mathcal{N}, {}^\forall Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}). \end{cases}$$

Then $\pi_{\mathcal{N}}$ is a $*$-representations of $\mathcal{N}$ in the Hilbert space $\mathcal{H}_{\mathcal{N}} \equiv \overline{\mathcal{D}(\pi_{\mathcal{N}})}$.

We denote by $P_{\mathcal{N}}$ the projection of $\mathcal{H}$ onto $\mathcal{H}_{\mathcal{N}} \equiv \overline{\mathcal{D}(\pi_{\mathcal{N}})}$. This projection $P_{\mathcal{N}}$ plays an important role in this reserch. First we have the following

Lemma 5.1.2. *$P_{\mathcal{N}}\mathcal{D} \subset \mathcal{D}^*(\pi_{\mathcal{N}})$ and $\pi^*_{\mathcal{N}}(X)P_{\mathcal{N}}\xi = P_{\mathcal{N}}X\xi$, ${}^\forall X \in \mathcal{N}$ and ${}^\forall \xi \in \mathcal{D}$.*

Definition 5.1.3. A map $\mathcal{E}$ of $\mathcal{M}$ into $\mathcal{L}^\dagger(\mathcal{D}(\pi_{\mathcal{N}}), \mathcal{H}_{\mathcal{N}})$ is said to be a *weak conditional-expectation* of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{N}$ if it satisfies

$$(AX\xi_0|Y\xi_0) = (\mathcal{E}(A)X\xi_0|Y\xi_0), \quad {}^\forall A \in \mathcal{M}, {}^\forall X, Y \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}).$$

For weak conditional-expectation we have the following

Theorem 5.1.4. *There exists a unique weak conditional-expectation $\mathcal{E}(\cdot|\mathcal{N})$ of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{N}$, and*

$$\mathcal{E}(A|\mathcal{N}) = P_{\mathcal{N}}A\lceil\mathcal{D}(\pi_{\mathcal{N}}), \quad {}^\forall A \in \mathcal{M}.$$

The weak conditional-expectation $\mathcal{E}(\cdot|\mathcal{N})$ of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{N}$ satisfies the following

(i) *$\mathcal{E}(\cdot|\mathcal{N})$ is linear,*

(ii) *$\mathcal{E}(\cdot|\mathcal{N})$ is hermitian, i.e., $\mathcal{E}(A|\mathcal{N})^\dagger = \mathcal{E}(A^\dagger|\mathcal{N})$, ${}^\forall A \in \mathcal{M}$,*

(iii) *$\mathcal{E}(X|\mathcal{N}) = X\lceil\mathcal{D}(\pi_{\mathcal{N}})$, ${}^\forall X \in \mathcal{N}$,*

(iv) *$\mathcal{E}(A^\dagger\square A|\mathcal{N}) \geq 0$, ${}^\forall A \in \mathcal{M}$ s.t. $A^\dagger\square A$ is well-defined ,*

(v) *$\mathcal{E}(A|\mathcal{N})^\dagger\square\mathcal{E}(A|\mathcal{N}) \leq \mathcal{E}(A^\dagger\square A|\mathcal{N}), {}^\forall A \in \mathcal{M}$ s.t. $A^\dagger\square A$ and $\mathcal{E}(A|\mathcal{N})^\dagger\square\mathcal{E}(A|\mathcal{N})$ are well-defined,*

(vi) *$\mathcal{E}(A|\mathcal{N})\square\pi_{\mathcal{N}}(X)$ is well-defined for any $A \in \mathcal{M}$ and $X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$, and $\mathcal{E}(A|\mathcal{N})\square\pi_{\mathcal{N}}(X) = \mathcal{E}(A\square X|\mathcal{N})$,*

(vii) *$\pi_{\mathcal{N}}(X)\square\mathcal{E}(A|\mathcal{N})$ is well-defined for any $A \in \mathcal{M} \cap R^{\mathrm{w}}(\mathcal{N})$ and ${}^\forall X \in \mathcal{N}$, and $\pi_{\mathcal{N}}(X)\square\mathcal{E}(A|\mathcal{N}) = \mathcal{E}(X\square A|\mathcal{N})$,*

(viii) *$\omega_{\xi_0}(\mathcal{E}(A|\mathcal{N})) = \omega_{\xi_0}(A)$, ${}^\forall A \in \mathcal{M}$.*

5.2 Unbounded conditional expectations for partial O*-algebras

Let $\mathcal{M}$ be a self-adjoint partial O*-algebra containing I on $\mathcal{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathcal{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N} \ni I$ is a partial O*-subalgebra of $\mathcal{M}$ satisfying (N):$(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}}$. We introduce unbounded conditional expectations of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{N}$.

Definition 5.2.1. A map $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ is said to be an *unbounded conditional expectation* of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{N}$ if

(i) the domain $D(\mathcal{E})$ of $\mathcal{E}$ is a $\dagger$-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$;

(ii) $\mathcal{E}$ is a projection; that is, it is hermitian ($\mathcal{E}(A)^\dagger = \mathcal{E}(A^\dagger), {}^\forall A \in D(\mathcal{E})$) and $\mathcal{E}(X) = X, {}^\forall X \in \mathcal{N}$;

(iii) $\mathcal{E}(A\square X) = \mathcal{E}(A)\square X$, ${}^\forall A \in D(\mathcal{E}), {}^\forall X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M})$,
$\mathcal{E}(X\square A) = X\square\mathcal{E}(A)$, ${}^\forall A \in \mathcal{D}(\mathcal{E}) \cap R^{\mathrm{w}}(\mathcal{N}), {}^\forall X \in \mathcal{N}$;

(iv) $\omega_{\xi_0}(\mathcal{E}(A)) = \omega_{\xi_0}(A), {}^\forall A \in D(\mathcal{E})$.

In particular, if $D(\mathcal{E}) = \mathcal{M}$, then $\mathcal{E}$ is said to be a *conditional expectation* of $(\mathcal{M}, \xi_0)$ w.r.t. $\mathcal{N}$.

For unbounded conditional expectations we have the following

Lemma 5.2.2. *Let $\mathcal{E}$ be an unbounded conditional expectation of $(\mathcal{M},\xi_0)$ w.r.t. $\mathcal{N}$. Then,*

$$\mathcal{E}(A)X\xi_0 = P_{\mathcal{N}}AX\xi_0 = \mathcal{E}(A|\mathcal{N})X\xi_0, \quad {}^{\forall}A \in D(\mathcal{E}), {}^{\forall}X \in \mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}).$$

Let $\mathfrak{E}$ be the set of all unbounded conditional expectations of $(\mathcal{M},\xi_0)$ w.r.t. $\mathcal{N}$. Then $\mathfrak{E}$ is an ordered set with the following order $\subset$:

$$\mathcal{E}_1 \subset \mathcal{E}_2 \text{ iff } D(\mathcal{E}_1) \subset \mathcal{D}(\mathcal{E}_2) \text{ and } \mathcal{E}_1(A) = \mathcal{E}_2(A), {}^{\forall}A \in D(\mathcal{E}_1).$$

Theorem 5.2.3. *There exists a maximal unbounded conditional expectation of $(\mathcal{M},\xi_0)$ w.r.t. $\mathcal{N}$, and it is denoted by $\mathcal{E}_{\mathcal{N}}$.*

5.3 Existence of conditional expectations for partial O*-algebras

Let $\mathcal{M}$ be a self-adjoint partial O*-algebra containing I on $\mathcal{D}$ in $\mathcal{H}$, $\xi_0 \in \mathcal{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and $\mathcal{N} \ni I$ a partial O*-subalgebra of $\mathcal{M}$ such that

(N) $(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is dense in $\mathcal{H}_{\mathcal{N}}$,

(N$_1$) $\mathcal{N}'_{\mathrm{w}}\widehat{\mathcal{D}}(\mathcal{N}) \subset \widehat{\mathcal{D}}(\mathcal{N})$,

(N$_2$)$(\mathcal{N} \cap R^{\mathrm{w}}(\mathcal{M}))\xi_0$ is essentially self-adjoint for $\mathcal{N}$,

(N$_3$) $\Delta''^{\,\mathrm{it}}_{\xi_0}(\mathcal{N}'_{\mathrm{w}})'\Delta''^{\,-\mathrm{it}}_{\xi_0} = (\mathcal{N}'_{\mathrm{w}})', {}^{\forall}t \in \mathbb{R}$, where Δ''_{ξ_0} is the modular operator for the full Hilbert algebra $(\mathcal{M}'_{\mathrm{w}})'\xi_0$.

Lemma 5.3.1. $\mathcal{D}(\mathcal{E}_{\mathcal{N}}) = \{A \in \mathcal{M}; P_{\mathcal{N}}A\xi_0 \in \mathcal{N}\xi_0\}$.

By Lemma 5.3.1 we have the following

Theorem 5.3.2. *Let $\mathcal{M}$ be a self-adjoint partial O*-algebra containing I on $\mathcal{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathcal{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N} \ni I$ is a partial O*-subalgebra of $\mathcal{M}$ satisfying (N), (N$_1$), (N$_2$) and (N$_3$). Then there exists a conditional expectation of $(\mathcal{M},\xi_0)$ w.r.t. $\mathcal{N}$ if and only if $P_{\mathcal{N}}\mathcal{M}\xi_0 = \mathcal{N}\xi_0$.*

It is important to investigate the scale of the domain of an unbounded conditional expectation. We consider the case of partial GW*-algebras.

Theorem 5.3.3. *Let $\mathcal{M}$ be a partial GW*-algebra on $\mathcal{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathcal{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N}$ be a partial GW*-subalgebra of $\mathcal{M}$satisfying (N), (N$_1$), (N$_2$) and (N$_3$). Then,*

$$\begin{aligned} D(\mathcal{E}_{\mathcal{N}}) \supset\ & \textit{linear span of } \{X\Box A; X \in \mathcal{N}, A \in (\mathcal{M}'_{\mathrm{w}})' \textit{ s.t.} \\ & \qquad X\Box A \textit{ and } X\Box\mathcal{E}''(A) \textit{ are well-defined}\} \\ \supset\ & \textit{linear span of } (\mathcal{M}'_{\mathrm{w}})' \textit{ and } \mathcal{N}. \end{aligned}$$

In particular, if $\mathcal{N}_{\mathcal{P}_{\mathcal{N}}}$ is a partial GW-algebra on $P_{\mathcal{N}}\mathcal{D}$, then $\mathcal{E}_{\mathcal{N}}$ is a conditional expectation of $(\mathcal{M},\xi_0)$ w.r.t. $\mathcal{N}$.*

Corollary 5.3.4. *Let $\mathcal{M}$ be a partial EW*-algebra on $\mathcal{D}$ in $\mathcal{H}$ and let $\xi_0 \in \mathcal{D}$ be a strongly cyclic and separating vector for $\mathcal{M}$ and suppose that $\mathcal{N}$ be a partial EW*-subalgebra of $\mathcal{M}$satisfying (N$_2$) and (N$_3$). Then,*

$$D(\mathcal{E}_{\mathcal{N}}) \supset \textit{ linear span of } \mathcal{M}_b\mathcal{N} \textit{ and } \mathcal{N}\mathcal{M}_b.$$

We consider the case of the well-known Segal L^p-space defined by τ.

Example 5.3.5. *Let $\mathcal{M}_0$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful finite trace τ. We denote by $L^p(\tau)$ the Banach space completion of $\mathcal{M}_0$ w.r.t. the norm*

$$\|A\|_p \equiv \tau(|A|^p)^{1/p}, \quad A \in \mathcal{M}_0.$$

Then

$$\mathcal{M}_0 \equiv L^\infty(\tau) \subset L^p(\tau) \subset L^2(\tau) \subset L^q(\tau) \subset L^1(\tau), \quad 1 \leq q \leq 2 \leq p < \infty.$$

Let $2 \leq p < \infty$. Here we define a $*$-representation π of $L^p(\tau)$ by

$$\pi(X)A = XA, \quad X \in L^p(\tau), A \in L^\infty(\tau).$$

Then $\mathcal{M} \equiv \pi(L^p(\tau))$ is a partial EW*-algebra on $L^\infty(\tau)$ in $L^2(\tau)$ with $\mathcal{M}_b = \pi(L^\infty(\tau))$ which is integrable, that is, $\overline{\pi(X^\dagger)} = \pi(X)^*$ for each $X \in L^p(\tau)$. Furthremore, $\pi(L^p(\tau))$ has a strongly cyclic and separating vector $\xi_0 \equiv \lambda_\tau(I)$, where I is an identity operator on $\mathcal{H}$. Let $\mathcal{N}_0$ be a von Neumann subalgebra of $\mathcal{M}_0$. We put

$$\mathcal{N} = \{\pi(X); X \in L^p(\tau), \pi(X)\lambda_\tau(I) \in L^p(\tau\lceil\mathcal{N}_0)\}, \quad 2 \leq p \leq \infty.$$

Then $\mathcal{N}$ is an integrable partial EW*-subalgebra of $\mathcal{M}$ satisfying (N_2) and (N_3) and $P_{\mathcal{N}}\mathcal{M}\xi_0 = \mathcal{N}\xi_0$. By Theorem 5.3.2, there exists a conditional expectation of $(\mathcal{M}, \xi_0)$.

Notes of Section 5. Unbounded conditional expectations for partial O*-algebras are due to [27].

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