Homogenization of Hamilton-Jacobi equations by an extended viscosity solution

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(Recieved November 30, 2005)

Abstract

We consider the periodic homogenization problem for Hamilton-Jacobi equation. We adapt the L-solution for the notion of weak solution. This enable us to treat the upper semicontinuous functions as the initial functions.

1 Introduction

Roughly speaking, the homogenization theory is the theory which tries to describe the macroscopic phenomenon from the point of view as the limit of microscopic phenomenon. The microscopic phenomenon may includes small periodic structure or small random structure.

In this note, we will point out that the standard theory is still valid for an extended viscosity solution.

First we give a short introduction to the theory of homogenization. Second, we shall introduce the notion of so-called L-solution and point out that we can follow the standard theory by using this extended solution.

2 What is the homogenization?

Let $H(x,p) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ be periodic with respect to x (that is, it holds $H(x + e_i, p) = H(x,p)$ for standard basis e_i, $i = 1, \ldots, n$), and satisfy the following coerciveness condition

$$H(x,p) \to \infty \quad \text{as } |p| \to \infty$$

uniformly with respect to $x \in \mathbb{R}^n$.

For $\varepsilon > 0$, we consider the Hamilton-Jacobi equations

$$\frac{\partial u^\varepsilon}{\partial t} + H\left(\frac{x}{\varepsilon}, D_x u^\varepsilon\right) = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^n \tag{1}$$

$$u^\varepsilon|_{t=0} = u_0(x) \in BUC(\mathbb{R}^n). \tag{2}$$

Here, $BUC(\mathbb{R}^n)$ denotes the space of all bounded uniformly continuous functions on $\mathbb{R}^n$, and $D_x u = (u_{x_1}, \ldots, u_{x_n})$ is the spatial gradient. We are interesting in the behaviour to the limit as $\varepsilon \to 0$ of the solution u^ε.

The following theorem is the first and the basic result due to Lions, Papanicolaou and Varadhan [3].

Theorem 2.1 *u^ε converges to some continuous function u, uniformly on $[0,T] \times \mathbb{R}^n$ $(\forall T > 0)$ as $\varepsilon \to 0$. Moreover, there exists $\overline{H}(p) \in C(\mathbb{R}^n)$ such that u is the viscosity solution of the following Hamilton-Jacobi equation:*

$$\frac{\partial u}{\partial t} + \overline{H}(D_x u) = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^n \tag{3}$$

$$u|_{t=0} = u_0(x) \in BUC(\mathbb{R}^n). \tag{4}$$

It is not obvious that how $\overline{H}$ is determined. We concern with the equation

$$H(y, p + D_y v) = \lambda$$

where p is the given parameter and λ is the parameter to be fixed. The problem in which we try to choose parameter λ to have the solution v, is called the *cell problem.*

The following result which is also due to Lions, Papanicolaou and Varadhan [3] is the key theorem in the homogenization theory.

Theorem 2.2 *For all $p \in \mathbb{R}^n$ there exists the unique $\lambda \in \mathbb{R}$ such that the equation*

$$H(y, p + D_y v) = \lambda$$

has the viscosity solution $v \in BUC(\mathbb{R}^n)$.

We denote this value λ as

$$\lambda = \overline{H}(p)$$

emphasising that λ depends on p. This $\overline{H}$ is called the *effective Hamiltonian.*

3 L-solution for Hamilton-Jacobi equations

The standard viscosity solution deals with continuous functions. However in some case in control theory, it appears discontinuous value functions which should be considered as viscosity solution to the associated Hamilton-Jacobi equations. There are lot of notions of extended viscosity solutions. In this note we employ L-solution which is introduced by Giga and Sato [1] and consider the homogenization theory for discontinuous initial conditions.

Let $u_0(x)$ be upper semicontinuous in $\mathbb{R}^n$ (we denote $u_0 \in USC(\mathbb{R}^n)$).

Define a closed set

$$K_0 = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leqq u_0(x)\},$$

and let

$$\psi_0(x, z) = -\min\{\mathrm{dist}((x, z), K_0), 1\}.$$

The function ψ_0, which is defined from the distant function from the set K_0, is a bounded uniformly continuous function on $\mathbb{R}^n \times \mathbb{R}$.

Note that ψ_0 is monotone decreasing with respect to z, and u_0 is the 0-level set of the function ψ_0:

$$\begin{aligned} u_0(x) &= \psi_0(x, 0) \\ &= \sup\{y \in \mathbb{R} \mid \psi_0(x, y) \geqq 0\}. \end{aligned}$$

The main idea of the L-solution is based on the level set approach.

Let us consider $u(t, x)$ as a 0-level set of some function $\psi(t, x, z)$, and transfer the equation with respect to ψ, and construct a viscosity solution associated with this level set equation. The solution of the original equation would be recovered by

$$u(t, x) = \sup\{z \mid \psi(t, x, z) \geqq 0\}.$$

This idea is justified by Giga and Sato [1].

Let us find the level set equation associated with Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + G(x, D_x u) = 0.$$

Since we consider $\psi(t, x, u(t, x)) = 0$, we have

$$\psi_t - \psi_z G\left(x, -\frac{\psi_x}{\psi_z}\right) = 0.$$

We put

$$F(y, \xi) = F(y, p, q) = \begin{cases} -qG\left(y, -\dfrac{p}{q}\right), & q < 0 \\ G_\infty(y, p), & q \geqq 0 \end{cases}$$

$$G_\infty(y, p) = \lim_{\lambda \downarrow 0} \lambda G\left(y, \frac{p}{\lambda}\right)$$

(we assume the existence of the limit in the second equation), and consider the equation

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + F(x, \psi_x, \psi_z) = 0 \\ &\psi(0, x, z) = \psi_0(x, z). \end{aligned}$$

It is verified that this equation is in the scope of the standard viscosity theory of continuous functions, and

$$u(t,x) = \sup\{z \mid \psi(t,x,z) \geqq 0\}$$

is an extended viscosity solution for original equation. This $u(t,x)$ is called the L-solution.

4 The main result

Now we return to our problem.

Hence we put

$$F(y,\xi) = F(y,p,q) = \begin{cases} -qH\left(y, -\dfrac{p}{q}\right), & q < 0 \\ H_\infty(y,p), & q \geqq 0 \end{cases}$$
$$H_\infty(y,p) = \lim_{\lambda \downarrow 0} \lambda H\left(y, \frac{p}{\lambda}\right)$$

(we assume the existence of the limit in the second equation), and consider the equation

$$\frac{\partial \psi^\varepsilon}{\partial t} + F\left(\frac{x}{\varepsilon}, \psi^\varepsilon_x, \psi^\varepsilon_z\right) = 0$$
$$\psi^\varepsilon(0,x,z) = \psi_0(x,z).$$

We can apply the standard homogenization theory to this level set equation. We can find the effective Hamiltonian $\overline{F}$, and the uniform limit ψ of ψ^ε which satisfies

$$\frac{\partial \psi}{\partial t} + \overline{F}(\psi_x, \psi_z) = 0$$
$$\psi|_{t=0} = \psi_0.$$

The cell problem for F is

$$H\left(y, \frac{p_1 + \psi^1_y}{q_1}\right) = \frac{\lambda(p_1,q_1)}{q_1}$$

for $q_1 < 0$. From this relation, it is easy to reduce that $\overline{F}(p_2,q_2) = \overline{F}(p_1,q_1)$ when $(p_1,q_1) = a(p_2,q_2)$ $(a > 0)$. Hence, we can verify from this property of $\overline{F}$, that there exists $\overline{H}$ such that

$$\overline{F}(p,q) = \begin{cases} -q\overline{H}\left(-\dfrac{p}{q}\right), & q < 0 \\ \overline{H}_\infty(p), & q \geqq 0 \end{cases}$$

and $u(t,x)$ defined by

$$u(t,x) = \sup\{y \in \mathbb{R} \mid \psi(t,x,y) \geqq 0\}$$

satisfies (3) and (4) in the sense of L-solution.

As the conclusion, we have proved the following result.

Theorem 4.1 *Assume that H satisfies the hypothesis of Theorem 2.1 and assume that the limit $H_\infty(y,p) = \lim_{\lambda\downarrow 0} \lambda H\left(y, \frac{p}{\lambda}\right)$ exists. Then their exists $\overline{H}(p) \in C(\mathbb{R}^n)$ and $u \in USC(\mathbb{R}^n)$ such that u is the L-solution of (3) with initial condition u_0.*

Example 4.1 Let us consider the Hamiltonian $H(x,p) = |p| - \cos 2\pi x$ which is described in [2]. In this case, we have

$$F(y,p,q) = \begin{cases} |p| + q\cos 2\pi y, & q < 0 \\ |p|, & q \geqq 0. \end{cases}$$

Hence, as described in [2], we have

$$\overline{F}(p,q) = \begin{cases} \max\{-q,\ |p|\}, & q < 0 \\ |p|, & q \geqq 0. \end{cases}$$

The effective Hamiltonian for H is $\overline{H}(p) = \max\{1,\ |p|\}$.

References

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