Bernstein's Inequality and Its Application

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1. Introduction

There are some types of inequalities which are known as Bernstein's Inequality. These ones of O. Takenouchi and T. Nishishiraho [3] and Y. Youichirou [4] are not best possible and one of A. Zigmund [5] is best possible. In the present paper, we state the essence of a proof of Bernstein's Inequality which is due to A. Zigmund [5] for the case of trigonometric polynomials with complex coefficients and using this result, we give the detail of the argument of a polarization constant which is given briefly in S. Dineen [1].

2. Preparation

Let n be a positive integer and k positive integers which satisfy that $1 \leq k \leq 2n$. Then let φ_{2n} be defined as follows;

$$\varphi_{2n}(t) = \frac{2k-1}{2n}\pi \quad (\frac{2k-1}{2n}\pi \leq t \leq \frac{2k+1}{2n}\pi)$$

We put

$$u_k = \frac{2k-1}{2n}\pi \quad (k = 1, 2, \ldots, 2n).$$

This $\varphi_{2n}(t)$ is a step function and each u_k is a jumping point.

For a fixed k which satisfies that $1 \leq k \leq 2n-1$,

$$\begin{aligned} &\textstyle\int_0^{2\pi} \cos kt d\varphi_{2n}(t) + i \int_0^{2\pi} \sin kt d\varphi_{2n}(t) \\ &= \textstyle\int_0^{2\pi} e^{ikt} d\varphi_{2n}(t) = \sum_{\ell=1}^{2n} e^{iku_\ell} \frac{\pi}{n} \\ &= \frac{\pi}{n} \textstyle\sum_{\ell=1}^{2n} e^{ik\frac{2\ell-1}{2n}\pi} = \frac{\pi}{n} e^{ik\frac{\pi}{2n}} \frac{1-e^{i2k\pi}}{1-e^{i\frac{k\pi}{n}}} = 0. \end{aligned}$$

Hence, if $1 \leq k \leq 2n-1$,

$$\int_0^{2\pi} \cos kt d\varphi_{2n}(t) = 0, \qquad \int_0^{2\pi} \sin kt d\varphi_{2n}(t) = 0.$$

Generally,

$$\cos kt \cdot \cos \ell t = \frac{1}{2}\{\cos(k+\ell)t + \cos(k-\ell)t\}$$

$$\sin kt \cdot \sin \ell t = \frac{1}{2}\{\cos(k-\ell)t - \cos(k+\ell)t\}$$

$$\sin kt \cdot \cos \ell t = \frac{1}{2}\{\sin(k+\ell)t + \sin(k-\ell)t.\}$$

Therefore, if $T(t)$ is a polynomial of degree $k(< 2n)$,

$$\int_0^{2\pi} T(t) d\varphi_{2n}(t) = 2\pi a_0,$$

where a_0 is a constant term of $T(t)$ and polynomial means a trigonometric polynomial.

Furthermore, if $1 \leq k \leq n-1$, we get

$$\int_0^{2\pi} d\varphi_{2n}(t) = 2\pi, \quad \int_0^{2\pi} \cos^2 kt d\varphi_{2n}(t) = \pi \ \text{ and } \ \int_0^{2\pi} \sin^2 kt d\varphi_{2n}(t) = \pi.$$

If $u_k = \frac{2k-1}{2n}\pi \quad (k = 1, 2, \ldots, 2n)$,

$$\cos nu_k = \cos(k - \frac{1}{2})\pi = 0 \quad \text{ and}$$

$$\sin nu_k = \sin(k - \frac{1}{2})\pi = -\cos k\pi = (-1)^{(k+1)}.$$

$$\int_0^{2\pi} \sin^2 nt d\varphi_{2n}(t) = \sum_{k=1}^{2n} \frac{\pi}{n} = 2\pi.$$

Then a sequence of the following functions

$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \ldots, \frac{1}{\sqrt{\pi}} \cos(n-1)t, \frac{1}{\sqrt{\pi}} \sin(n-1)t, \frac{1}{\sqrt{2\pi}} \sin nt$

is an orthonormal system on interval $(0, 2\pi)$ with respect to weight $d\varphi_{2n}(t)$.

It is easily seen the following

Lemma.

If we put

$$S(t) = \frac{1}{2}a_0 + \sum_{\nu=1}^{n-1} a_\nu \cos \nu t + \sum_{\nu=1}^{n-1} b_\nu \sin \nu t + \frac{1}{2} b_n \sin nt,$$

for any complex numbers a_ν and b_ν , then

$$a_\nu = \frac{1}{\pi} \int_0^{2\pi} S(t) \cos \nu t d\varphi_{2n}(t) \quad (\nu = 0, 1, 2, \ldots, n-1),$$

$$b_\nu = \frac{1}{\pi} \int_0^{2\pi} S(t) \sin \nu t d\varphi_{2n}(t) \quad (\nu = 1, 2, \ldots, n).$$

If $1 \leq \nu \leq n-1$,

$a_\nu \cos \nu x + b_\nu \sin \nu x$

$= \frac{1}{\pi} \int_0^{2\pi} S(t)(\cos \nu x \cos \nu t + \sin \nu x \sin \nu t) d\varphi_{2n}(t)$

$= \frac{1}{\pi} \int_0^{2\pi} S(t) \cos \nu(t-x) d\varphi_{2n}(t).$

Since $\cos nu_k = 0$,

$$\int_0^{2\pi} S(t) \frac{1}{2} \cos nt \cos nx d\varphi_{2n}(t) = 0.$$

Therefore,
$S(x)=\frac{1}{\pi}\int_0^{2\pi}S(t)\{\frac{1}{2}+\sum_{\nu=1}^{n-1}\cos\nu(t-x)+\frac{1}{2}\sin nt\sin nx+\frac{1}{2}\cos nt\cos nx\}d\varphi_{2n}(t)$
$=\frac{1}{\pi}\int_0^{2\pi}S(t)\{\frac{1}{2}+\sum_{\nu=1}^{n-1}\cos\nu(t-x)+\frac{1}{2}\cos n(t-x)\}d\varphi_{2n}(t).$
We notice the following

$$\frac{1}{2}+\sum_{\nu=1}^{n}\cos\nu v=\frac{\sin(n+\frac{1}{2})v}{2\sin\frac{1}{2}v}\quad\text{and}$$

$$\frac{1}{2}+\sum_{\nu=1}^{n-1}\cos\nu v+\cos nv=\frac{\sin nv}{2\tan\frac{1}{2}v}.$$

We put

$$\begin{aligned}D_n^*(t-x)&=\frac{1}{2}+\sum_{\nu=1}^{n-1}\cos\nu(t-x)+\frac{1}{2}\cos n(t-x)\\&=\frac{\sin n(t-x)}{2\tan\frac{1}{2}(t-x)}.\end{aligned}$$

$D_n^*(v)$ is called the modified Dirichlet kernel. By the way, a general polynomial of degree n is of the form

$$\begin{aligned}T(x)&=S(x)+a_n\cos nx\\&=a_n\cos nx+\frac{1}{\pi}\int_0^{2\pi}S(t)D_n^*(t-x)d\varphi_{2n}(t).\end{aligned}$$

Since $\int_0^{2\pi}\cos ntD_n^*(t-x)d\varphi_{2n}(t)=0$,

$$\begin{aligned}T(x)&=a_n\cos nx+\frac{1}{\pi}\int_0^{2\pi}S(t)D_n^*(t-x)d\varphi_{2n}(t)\\&=a_n\cos nx+\frac{1}{\pi}\int_0^{2\pi}T(t)D_n^*(t-x)d\varphi_{2n}(t).\end{aligned}$$

$$T'(x)=-na_n\sin nx+\frac{1}{\pi}\int_0^{2\pi}T(t)\frac{d}{dx}D_n^*(t-x)d\varphi_{2n}(t).$$

$$\begin{aligned}\frac{d}{dx}D_n^*(t-x)&=\frac{d}{dx}\frac{\sin n(t-x)}{2\tan\frac{1}{2}(t-x)}\\&=\frac{n\cos n(x-t)}{2\tan\frac{1}{2}t}-\frac{\sin n(x-t)}{4\sin^2\frac{1}{2}(x-t)}.\end{aligned}$$

$$\begin{aligned}T'(0)&=\frac{1}{\pi}\int_0^{2\pi}T(t)\left(\frac{n\cos nt}{2\tan\frac{1}{2}t}+\frac{\sin nt}{4\sin^2\frac{1}{2}t}\right)d\varphi_{2n}(t)\\&=\frac{1}{\pi}\sum_{k=1}^{2n}T(u_k)\frac{\sin nu_k}{(2\sin\frac{1}{2}u_k)^2}\\&=\frac{1}{\pi}\sum_{k=1}^{2n}T(u_k)\frac{(-1)^{k+1}}{(2\sin\frac{1}{2}u_k)^2}\\&=\sum_{k=1}^{2n}T(u_k)(-1)^{k+1}\alpha_k,\end{aligned}$$

where $\alpha_k=\frac{1}{\pi(2\sin\frac{1}{2}u_k)^2}$.
Now we put $\hat{T}(x)=T(\theta+x)$.

$$\frac{d}{dx}T(\theta+x)|_{x=0}=T'(\theta)=\hat{T}'(0)=\sum_{k=1}^{2n}\hat{T}(u_k)(-1)^{k+1}\alpha_k.$$

Hence

$$T'(\theta)=\sum_{k=1}^{2n}T(\theta+u_k)(-1)^{k+1}\alpha_k\cdots(1).$$

Thus we get

$$|T'(\theta)|\leq\sum_{k=1}^{2n}\alpha_k|T(\theta+u_k)|.$$

Now we consider the next special case:

$$S(x)=\sin nx.$$

Then

$$S'(x)=n\cos nx \text{ and } S'(0)=n.$$

By above (1),

$$S'(0)=T'(0)=\sum_{k=1}^{2n}S(u_k)(-1)^{k+1}\alpha_k=\sum_{k=1}^{2n}\alpha_k,$$

since $S(u_k)=(-1)^{k+1}$. Hence $\sum_{k=1}^{2n}\alpha_k=n$.
If $|T(x)|\leq M$, we obtain

$$|T'(\theta)|\leq M\sum_{k=1}^{2n}\alpha_k\leq Mn.$$

Thus we get the following

Proposition 1 (Bernstein's Inequality).
If a polynomial $T(x)$ of order n satisfies $|T(x)|\leq M$ for all x, then $|T'(x)|\leq Mn$.

3. An application of Bernstein's Inequality

Let $\mathbf{C}$ be the complex field and H a comlex Hilbert space. Let $\check{p}:H^n\to\mathbf{C}$ be n-linear mapping and $p(x)=\check{p}(x,x,\cdots,x)$ the corresponding homogeneous polynomial of degree n. We put

$$\overline{B}_H=\{x\in H;\|x\|\leq 1\},$$

and

$$\mathcal{P}(^nH)=\{p;p \text{ is a homogeneous polynomial of degree } n\},$$

When $p\in\mathcal{P}(^nH)$, we put

$$\|p\|=\sup\{|p(x)|;x\in\overline{B}_H\},$$

$$\|\check{p}\|=\sup\{|\check{p}(x_1,\cdots,x_n)|;x_j\in\overline{B}_H,(j=1,\cdots,n)\},$$

and

$$c(n,H)=\inf\{M; \| \check{p} \| \le M \| p \| \text{ for any } p \in \mathcal{P}(^nH)\}.$$

This $c(n,H)$ is called the n^{th} polarization constant of the Hilbert space H and $c(E) := \limsup_{n\to\infty} c(n,H)^{\frac{1}{n}}$ is called the polarization constant of the space H. It is clear that $c(n,H) \ge 1$, since $\| p \| \le \| \check{p} \|$.

Now we put

$$\sigma = \begin{cases} 1 & \text{if } (x,y)=0 \\ \frac{(x,y)}{|(x,y)|} & \text{if } (x,y)\neq 0. \end{cases}$$

If $x, y \in \overline{B}_H$, then $x\cos\theta + i\sigma y\sin\theta \in \overline{B}_H$, since

$$\begin{aligned} &\| x\cos\theta + i\sigma y \sin\theta \|^2 \\ &= (x\cos\theta + i\sigma y\sin\theta, x\cos\theta + i\sigma y\sin\theta) \\ &\le \max\{\| x \|^2, \| y \|^2\}. \end{aligned}$$

When we put

$T_n(\theta) = p(x\cos\theta + i\sigma y\sin\theta)$

$= \check{p}(x\cos\theta + i\sigma y\sin\theta, \cdots, \cdots, x\cos\theta + i\sigma y\sin\theta)$ for $x, y \in \overline{B}_H$,

$T_n(\theta)$ is a trigonometric polynomial of degree n with complex coefficients.

$x\cos\theta + i\sigma y\sin\theta$

$= x\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\theta^{2k} + i\sigma y\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\theta^{2k+1}$

$= x + i\sigma y\theta + \sum_{k\ge 2} a_k\theta^k$,

where a_k are some complex numbers.

$T_n(\theta) = p(x + i\sigma y\theta + \sum_{k\ge 2} a_k\theta^k)$

$= \check{p}(x + i\sigma y\theta + \sum_{k\ge 2} a_k\theta^k, \cdots, x + i\sigma y\theta + \sum_{k\ge 2} a_k\theta^k)$

$= \check{p}(x,\cdots,x) + n\theta\check{p}(x,\cdots,x,i\sigma y) + \sum_{k\ge 2} b_k\theta^k$,

where b_k are some complex numbers.

$$T_n'(0) = \frac{d}{d\theta}T_n(\theta)|_{\theta=0} = n\check{p}(x,\cdots,x,i\sigma y) = i\sigma n\check{p}(x,\cdots,x,y_n),$$

where $y_n = y$.

By Bernstein's Inequality,

$$|T_n'(0)| \le n\max|T_n(\theta)| \le n \| p \| .$$

Since $|i\sigma| = 1$,

$$|\check{p}(x,\cdots,x,y_n)| \le \| p \| .$$

For a fixed y_n, we put

$$p_{n-1}(x) = \check{p}(x,\cdots,x,y_n).$$

Then we can look upon $p_{n-1}(x)$ as homogeneous polynomial of degree $n-1$ and

$$\| p_{n-1} \| = \sup\{|\check{p}_{n-1}(x,\cdots,x,y_n)|; x \in \overline{B}_H\} \leq \| p \| .$$

Similarly we get

$$|\check{p}(x,\cdots,x,y_{n-1},y_n)| \leq \| p_{n-1} \| .$$

After all we get

$$|\check{p}(x,y_2,\cdots,y_{n-1},y_n)| \leq \| p \| .$$

Hence $\| \check{p} \| \leq \| p \|$. Thus we have the following

Proposition 2.

If H is a complex Hilbert space, then $c(n,H) = 1$.

Remark.

When $T_n(x)$ is a trigonometric polynomial of degree n with $|T_n(x)| \leq M$ for any x, the inequality

$$|T_n'(x)| \leq 2nM$$

is also called Bernstein's Inequality. This is not best possible. However, in this case, we don't need Stieltjes integral for the proof, because

$$T_n'(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} T(x-t)2n\sin ntF_{n-1}(t)dt,$$

where

$$F_n(t) = \sum_{j=-n}^{n}(1 - \frac{|j|}{n+1})e^{ijt}$$

which is called Fejér kernel.

References

[1] S. Dineen, *Complex Analysis on Infinite Dimensional Analysis* , Springer (1998)

[2] L. A. Harris, *Bernstein's Polynomial Inequalities and Functional Analysis,* Irish Math. Soc. Bull. **36** (1996), 19-33

[3] O. Takenouchi and T. Nishishiraho, *Approximarion Theory* (in Japanese), Baifukan, (1986)

[4] Y. Youichirou, *Real Functions and Fourier Analysis* **2** (in japanese), Iwanami, (1998)

[5] A. Zigmund, *Trigonometric Series,* Cambridge Press (1959)