Distortion and Characterization Theorems for Generalized Fractional Integration Operators Involving H-Function in Subclasses of Univalent Functions

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1. Introduction

Let $A(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$, and let $S(n)$ denote the subclass of $A(n)$ of *univalent functions* in U. The so-called subclass of functions with negative coefficients is also often considered, denoted by $T(n) \subset S(n)$, of the functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ k = n+1, n+2, \ldots). \tag{2}$$

We consider some mapping, distortion and characterization properties of the operators of the generalized fractional calculus involving Fox's H-functions (Kiryakova [8]) in the classes $A(n), S(n), T(n)$ and their subclasses of the so-called *starlike and convex functions of order* α $(0 \leqq \alpha < 1)$.

In this way we extend our previous results (see Kiryakova, Saigo and Owa [10]) related to the operators of generalized fractional calculus involving Meijer's G-functions, and including the hypergeometric fractional integration operators by Saigo ([21]–[23], [31]) and Hohlov ([3], [4]), the Appell's F_3-function operators by Saigo ([24], [25]) and most of the classical integral operators considered in classes of univalent functions by various authors.

2. Preliminaries

We remind first the definitions of some special functions referred to in this paper.

By a *Fox's H-function* we mean a generalized hypergeometric function defined by means of the Mellin–Barnes type contour integral

$$H_{p,q}^{m,n}\left[\sigma \,\middle|\, \begin{matrix} (a_k, A_k)_1^p \\ (b_k, B_k)_1^q \end{matrix}\right] = \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\prod_{k=1}^{m} \Gamma(b_k - sB_k) \prod_{j=1}^{n} \Gamma(1 - a_j + sA_j)}{\prod_{j=n+1}^{p} \Gamma(a_j - sA_j) \prod_{k=m+1}^{q} \Gamma(1 - b_k + sB_k)} \sigma^s \, ds, \tag{3}$$

where $\mathfrak{L}$ is a suitable contour in $\mathbb{C}$, the orders (m,n,p,q) are integers $0 \leqq m \leqq q$, $0 \leqq n \leqq p$ and the parameters $a_j \in \mathbb{R}$, $A_j > 0$ $(j = 1,\ldots,p)$, $b_k \in \mathbb{R}$, $B_k > 0$ $(k = 1,\ldots,q)$ are such that $A_j(b_k + l) \neq B_k(a_j - l' - 1)$ $(l, l' = 0,1,2,\ldots)$. For various type of contours and conditions for existence and analyticity of function (3) in disks $\subset \mathbb{C}$ whose radii are $\rho = \prod_{j=1}^{p} A_j^{-A_j} \prod_{k=1}^{q} B_k^{B_k} > 0$ of the H-functions, one can see in [5], [8, App.], [15], [28], etc.

When $A_1 = \ldots = A_p = B_1 = \ldots = B_q = 1$, (3) turns into the more popular *Meijer's G-function* (see [2, Vol.1, Ch.5], [5], [8], [15]). The G- and H-functions encompass almost all the elementary and special functions and this makes the knowledge on them very useful. Observe that the generalized hypergeometric functions ${}_pF_q$, and thus, most of the classical special functions, are special cases of the G-function:

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\sigma) = \frac{\prod\limits_{k=1}^{q}\Gamma(b_k)}{\prod\limits_{j=1}^{p}\Gamma(a_j)} G^{1,p}_{p,q+1}\left[-\sigma \left| \begin{matrix} 1-a_1,\ldots,1-a_p \\ 0,1-b_1,\ldots,1-b_q \end{matrix} \right.\right]. \tag{4} $$

On the other side, the *Mittag-Leffler functions* $E_{\rho,\mu}$ (appearing as solutions of fractional order differential and integral equations) and the *Wright's generalized hypergeometric functions* ${}_p\Psi_q$ with irrational $A_j, B_k > 0$, give examples of *H-functions, not reducible to G-functions*:

$$ {}_p\Psi_q\left(\begin{matrix} (a_1,A_1),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_q,B_q) \end{matrix} ;\sigma \right) = \sum_{k=0}^{\infty} \frac{\Gamma(a_1+kA_1)\ldots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\ldots\Gamma(b_q+kB_q)} \frac{\sigma^k}{k!} $$

$$ = H^{1,p}_{p,q+1}\left[-\sigma \left| \begin{matrix} (1-a_1,A_1),\ldots,(1-a_p,A_p) \\ (0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q) \end{matrix} \right.\right]. \tag{5} $$

However, for $A_1 = \ldots = A_p = B_1 = \ldots = B_q = 1$,

$$ {}_p\Psi_q\left(\begin{matrix} (a_1,1),\ldots,(a_p,1) \\ (b_1,1),\ldots,(b_q,1) \end{matrix} ;\sigma \right) = \frac{\prod\limits_{j=1}^{p}\Gamma(a_j)}{\prod\limits_{k=1}^{q}\Gamma(b_k)}\, {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\sigma). \tag{6} $$

In the scheme of the typical H-functions we have recently included and studied also multi-index analogues of $E_{\rho,\mu}$, called *multiindex Mittag-Leffler functions* (see Kiryakova [9]).

Using as kernel-function a Meijer's G-function, and more generally - a Fox's H-function of peculiar order $(m,0,m,m)$, a *generalized fractional calculus* has been developed in Kiryakova [8] that includes as special cases almost all the known operators of fractional integration and differentiation studied by many authors. Especially, even the particular case with a G-function kernel, has been shown (Kiryakova [8, Ch.5], Kiryakova, Saigo and Owa [10], Kiryakova, Saigo and Srivastava [11]) to encompass most of the integro-differential operators already popular in univalent functions theory.

Let $m \geqq 1$ be an integer and $\delta_i \geqq 0, \gamma_i \in \mathbb{R}, \beta_i > 0$ $(i = 1,\ldots,m)$. We consider $\delta = (\delta_1,\ldots,\delta_m)$ as a *multiorder of fractional integration*, resp., $\gamma = (\gamma_1,\ldots,\gamma_m)$ as multiweight, $\beta = (\beta_1,\ldots,\beta_m)$ as additional parameter. The integral operators defined as follows:

$$ I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \begin{cases} \displaystyle\int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_i+\delta_i+1-1/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i+1-1/\beta_i, 1/\beta_i)_1^m \end{matrix} \right.\right] f(z\sigma)d\sigma, & \text{if } \displaystyle\sum_{i=1}^{m}\delta_i > 0, \\ f(z), \quad \text{if} \quad \delta_1 = \delta_2 = \ldots = \delta_m = 0, \end{cases} \tag{7} $$

are said to be *multiple* (m*-tuple*) *Erdélyi–Kober fractional integration operators* and more generally, all the operators of the form

$$If(z) = z^{\delta_0} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) \quad \text{with} \quad \delta_0 \geqq 0$$

are called briefly *generalized* (m*-tuple*) *fractional integrals.*

The corresponding generalized fractional derivatives are denoted by $D^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$ and defined by means of explicit differintegral expressions (see [8]), similarly to the idea for the classical Riemann-Liouville derivative. For $m = 1$, operators (7) turn into the *Erdélyi–Kober fractional integrals* $I^{\gamma,\delta}_{\beta}$, widely used in the applied mathematical analysis (see [8], [26]) and to the *classical Riemann-Liouville fractional integrals* R^{δ}:

$$I^{\gamma,\delta}_{\beta} f(z) = \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} \sigma^{\gamma} f(z\sigma^{1/\beta}) d\sigma \quad (\delta > 0, \gamma \in \mathbb{R}, \beta > 0), \tag{8}$$

$$R^{\delta} f(z) = z^{\delta} \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(z\sigma) d\sigma = z^{\delta} I^{0,\delta}_{1} f(z) \quad (\delta > 0), \tag{9}$$

namely:

$$R^{\delta} f(z) = z^{\delta} I^{0,\delta}_{1,1} f(z), \quad I^{\gamma,\delta}_{\beta} f(z) = I^{\gamma,\delta}_{1,1} f(z)$$

for $m = 2$ - into the hypergeometric fractional integrals (Love, Saigo, Hohlov, etc.), and for various other special choices of $m \geqq 1$ and of parameters, to many other generalized integration and differentiation operators, used in analysis, including in univalent functions theory, integral transforms and special functions, differential and integral equations, etc.

The main feature of the generalized (m-tuple) fractional integrals is that single integrals (7) involving H-functions (or G-functions in the simpler case of equal $\beta_i = \beta > 0, i = 1, \ldots, m$) can be equivalently represented by means of *commutative compositions of finite number m of Erdélyi–Kober integrals* (8), namely: in the case considered here, for $\gamma_i \geqq -1$, $\delta_i \geqq 0$, $\beta_i > 0$ $(i = 1, \ldots, m)$,

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \left[\prod_{i=1}^{m} I^{\gamma_i,\delta_i}_{\beta_i}\right] f(z) = \int_0^1 \ldots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-\sigma_i)^{\delta_i - 1} \sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right] f\left(z\sigma_1^{1/\beta_1} \ldots \sigma_m^{1/\beta_m}\right) d\sigma_1 \ldots d\sigma_m. \tag{10}$$

If some of the δ_i are zeros: $\delta_1 = \ldots = \delta_s = 0, 1 \leqq s \leqq m$, the corresponding multipliers are identity operators ($I^{\gamma_i,\delta_i}_{\beta_i} = I$) and the multiplicity of (7), (10) reduces from m to $m - s$ (the same for the order of the kernel H-functions). Decomposition (10) is the key to numerous applications of (7), arising from the simple but quite effective tools of the G- and H-functions.

A detailed theory, called *generalized fractional calculus*, and an analogue of the classical fractional calculus and its different applications are proposed in [8]. Here we consider *some mapping properties of operators* (7) *in classes of analytic functions in the unit disk* $U = \{z : |z| < 1\}$.

Using only the simple properties of Fox's H-function ([8, App.], [15], [28]), one easily obtains the following.

Lemma 0. *For* $\delta_i \geqq 0, \gamma_i \in \mathbb{R}, \beta_i > 0$ $(i = 1, \ldots, m)$, *and each* $p > \max\limits_{1 \leqq i \leqq m} [-\beta_i(\gamma_i + 1)]$,

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\{z^p\} = \lambda_p\, z^p \quad \text{with} \quad \lambda_p = \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + p/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + p/\beta_i)} > 0. \tag{11}$$

Then the conditions

$$\delta_i \geqq 0, \quad \gamma_i \geqq -1, \quad \beta_i > 0 \quad (i = 1, \ldots, m) \tag{12}$$

ensure that (11) *holds for each* $p \geqq 0$.

Proof. To evaluate the $I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$-image of an arbitrary power function $f(z) = z^p$, we use an extension of known integral formulas for the H-functions, namely formula [8, App., (E.21)]:

$$\int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (a_i, C_i)_1^m \\ (b_i, C_i)_1^m \end{matrix}\right] d\sigma = \prod_{i=1}^{m} \frac{\Gamma(b_i + C_i)}{\Gamma(a_i + C_i)} \quad \text{for} \quad a_i > b_i > -C_i \quad (i = 1, \ldots, m).$$

Then, according to the well known H-function's property (see [8, App., (E.9)]), we obtain

$$\begin{aligned} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\{z^p\} &= \int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_i + \delta_i + 1 - 1/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + 1 - 1/\beta_i, 1/\beta_i)_1^m \end{matrix}\right] z^p \sigma^p\, d\sigma \\ &= z^p \int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_i + \delta_i + 1 + (p-1)/\beta_i)_1^m \\ (\gamma_i + 1 + (p-1)/\beta_i)_1^m \end{matrix}\right] d\sigma \\ &= z^p \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + p/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + p/\beta_i)} = \lambda_p z^p, \end{aligned}$$

where the conditions $\gamma_i + \delta_i + p/\beta_i > \gamma_i + p/\beta_i > -1$ $(i = 1, \ldots, m)$ are ensured by $\delta_i \geqq 0$ and $\gamma_i > -1 - p/\beta_i$ $(i = 1, \ldots, m)$, i.e. $p > \max\limits_i [-\beta_i(\gamma_i + 1)]$. To have (11) for all z^p $(p \geqq 0)$ it suffices to ask $\gamma_i \geqq -1$. ■

In view of formula (11), for considering functions in the classes $A(n)$, $S(n)$, $T(n)$, it is suitable to *normalize* the operators (7) by the multiplier constant $[\lambda_1]^{-1}$ $(p = 1)$. Therefore, further we consider the generalized fractional integrals (using the same name for the normalized version, but stressing this fact by an additional "tilde" in the denotation: $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} := [\lambda_1]^{-1} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$,

$$\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) := \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + 1 + 1/\beta_i)}\; I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z). \tag{13}$$

Thus, from Lemma 0 and the more general results in [8, Ch.5, §5.5], [12, Th.1], we can easily obtain the following:

Theorem 1. *Under the parameters' conditions* (12) :

$$\delta_i \geqq 0, \quad \gamma_i \geqq -1, \quad \beta_i > 0 \quad (i = 1, \ldots, m)$$

the generalized fractional integral $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$ *maps the class* $A(n)$ *into itself, and the image of a power series* (1) *has the form*

$$\widetilde{I}f(z) = \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left\{z + \sum_{k=n+1}^{\infty} a_k z^k\right\} = z + \sum_{k=n+1}^{\infty} \theta(k)\, a_k z^k \in A(n) \tag{14}$$

with multipliers' sequence:

$$\theta(k) = \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + k/\beta_i)\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + k/\beta_i)\Gamma(\gamma_i + 1 + 1/\beta_i)} > 0 \quad (k = n+1, n+2, \ldots). \tag{15}$$

Proof. First we need to establish the fact that

$$\lim_{k\to\infty} |\theta(k)|^{1/k} = 1 \tag{16}$$

for $\theta(k) = \lambda_k/\lambda_1$. Denote, for brevity in the proofs of this and the next theorems,

$$\begin{aligned} &a_i = \gamma_i + \delta_i + 1, \quad b_i = \gamma_i + 1, \quad \kappa_i = k/\beta_i, \\ &c_i = a_i + (n+1)/\beta_i, \quad d_i = b_i + (n+1)/\beta_i \quad (i = 1, \ldots, m;\ k = n+1, \ldots), \end{aligned} \tag{17}$$

from where and from (12) evidently,

$$a_i \geqq b_i, \quad c_i \geqq d_i \quad \text{and} \quad \kappa_i \to \infty \quad (i = 1, \ldots, m) \quad \text{as} \quad k \to \infty.$$

The known asymptotics

$$\frac{\Gamma(b+\kappa)}{\Gamma(a+\kappa)} \sim \kappa^{b-a} \quad \text{as} \quad \kappa \to \infty$$

yields

$$\left[\frac{\Gamma(b_i+\kappa)}{\Gamma(a_i+\kappa)}\right]^{1/k} \sim \left(\kappa_i^{-\delta_i}\right)^{1/k} = \left(k^{1/k}\right)^{-\delta_i} \cdot \left(\beta_i^{\delta_i}\right)^{1/k}$$

and the limit equalities $\lim_{k\to\infty} k^{1/k} = 1$, $\lim_{k\to\infty} q^{1/k} = 1$ for $q =$const give:

$$\lim_{\kappa_i\to\infty}\left[\frac{\Gamma(b_i+\kappa_i)}{\Gamma(a_i+\kappa_i)}\right]^{1/k} = 1 \quad \text{and} \quad \lim_{k\to\infty}\left[\frac{\Gamma(a_i+1/\beta_i)}{\Gamma(b_i+1/\beta_i)}\right]^{1/k} = 1 \quad (i = 1, \ldots, m).$$

We have then

$$\lim_{k\to\infty} |\theta(k)|^{1/k} = \lim_{k\to\infty} \prod_{i=1}^{m}\left[\frac{\Gamma(b_i+\kappa_i)}{\Gamma(a_i+\kappa_i)}\right]^{1/k}\left[\frac{\Gamma(a_i+1/\beta_i)}{\Gamma(b_i+1/\beta_i)}\right]^{1/k} = 1,$$

which is (16).

Under the assumptions of the theorem, Lemma 0 guarantees that

$$\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\{z\} = z \quad \text{and} \quad \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\{z^k\} = \frac{\lambda_k}{\lambda_1} z^k = \theta(k)\, z^k$$

and term-by-term integration of power series (1) gives series (14). By virtue of the Cauchy–Hadamard formula, the radius of convergence of the first series, as an analytic function in the unit disk, is $R = \left\{ \overline{\lim\limits_{k\to\infty}} |a_k|^{1/k} \right\}^{-1} \geqq 1$, and that of the latter series is calculated by

$$\widetilde{R} = \left\{ \overline{\lim_{k\to\infty}} |a_k|^{1/k} \cdot |\theta(k)|^{1/k} \right\}^{-1},$$

therefore $\widetilde{R} \geqq 1$ and the image $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)$ given by series (14) is analytic in the unit disc, too. Note that due to positiveness of the multipliers $\theta(k)$, the operator $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$ preserves the map of the series with positive (like in $A(n)$) and negative (like in $T(n)$) coefficients into series of the same kind. ■

The *Hadamard product* (*convolution*) of two analytic functions in U: $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum\limits_{k=0}^{\infty} b_k z^k$ is defined by

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k.$$

Theorem 2. *In the class* $A(n)$ *the generalized fractional integral* (13) *can be represented by the Hadamard product*

$$\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = (h * f)(z), \tag{18}$$

where the function $h(z) \in A(n)$ *is expressed by the Wright generalized hypergeometric function* (5)

$$\begin{aligned} h(z) &= z + \sum_{k=n+1}^{\infty} \theta(k) z^k \\ &= z + \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + 1 + 1/\beta_i)} z^{n+1} \; {}_{m+1}\Psi_m \left(\begin{matrix} (1,1), (\gamma_i + 1 + (n+1)/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + \delta_i + 1 + (n+1)/\beta_i, 1/\beta_i)_1^m z \end{matrix} ; z \right). \end{aligned} \tag{19}$$

Proof. Changing the index of summation and using the short denotations in (17), we get

$$\begin{aligned} h(z) &= z + \sum_{k=n+1}^{\infty} \theta(k) z^k = z + \frac{z^{n+1}}{\lambda_1} \sum_{j=0}^{\infty} \lambda_{j+(n+1)}\, z^j \\ &= z + \frac{z^{n+1}}{\lambda_1} \sum_{j=0}^{\infty} \left\{ \Gamma(1+j) \prod_{i=1}^{m} \frac{\Gamma(d_i + j/\beta_i)}{\Gamma(c_i + j/\beta_i)} \right\} \frac{z^j}{j!} \\ &= z + \frac{z^{n+1}}{\lambda_1} \; {}_{m+1}\Psi_m \left(\begin{matrix} (1,1), (d_1, 1/\beta_1), \dots, (d_m, 1/\beta_m) \\ (c_1, 1/\beta_1), \dots, (c_m, 1/\beta_m) \end{matrix} ; z \right), \end{aligned}$$

which gives (19). ■

Corollary 1. *For $n = 1$ in the classes A, S and T, the representation of the "convolution function" $h(z)$ in (19) simplifies as:*

$$h(z) = z + \frac{z^2}{\lambda_1} {}_{m+1}\Psi_m \left(\begin{array}{c} (1,1), (\gamma_i + 1 + 2/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + \delta_i + 1 + 2/\beta_i, 1/\beta_i)_1^m \end{array} ; z \right). \tag{20}$$

We now consider the case when all $\beta_i = \beta > 0$ $(i = 1, \ldots, m)$, and especially for shortness of denotations, it is taken $\beta = 1$, for the generalized fractional integrals with Meijer's G-function in the kernel,

$$\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) = \widetilde{I}_{(1,1,\ldots,1),m}^{(\gamma_i),(\delta_i)} f(z) = \frac{1}{\lambda_1} \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{array}{c} (\gamma_i + \delta_i)_1^m \\ (\gamma_i)_1^m \end{array} \right. \right] f(z\sigma) d\sigma. \tag{21}$$

Corollary 2. *For the operator (21) the simpler representations of multipliers' sequence $\theta(k)$ and convolution function $h(z)$ take the forms*

$$\theta(k) = \prod_{i=1}^{m} \frac{(\gamma_i + 2)_{k-1}}{(\gamma_i + \delta_i + 2)_{k-1}} > 0 \quad (k = n+1, n+2, \ldots) \tag{22}$$

with $(a)_k = \Gamma(a+k)/\Gamma(a)$ denoting the known Pochhammer symbol, and

$$h(z) = z + \prod_{i=1}^{m} \frac{(\gamma_i + 2)_n}{(\gamma_i + \delta_i + 2)_n} z^{n+1} {}_{m+1}F_m \left(\begin{array}{c} 1, (\gamma_i + 2 + n)_1^m \\ (\gamma_i + \delta_i + 2 + n)_1^m \end{array} ; z \right). \tag{23}$$

For $n = 0$ (i.e. in the classes A, S, T), $h(z)$ simplifies to a ${}_{m+1}F_m$-generalized hypergeometric function:

$$h(z) = z + z^2 {}_{m+1}F_m \left(\begin{array}{c} 1, (\gamma_i + 2)_1^m \\ (\gamma_i + \delta_i + 2)_1^m \end{array} ; z \right). \tag{24}$$

Many special cases of operators (13), or of their modified form $cz^{\delta_0} \widetilde{I}_{(\beta_i),m}^{(\gamma_i),(\delta_i)} f(z)$ with $c = \text{const}$ and $\delta_0 \geqq 0$, especially in the case with kernel-function reducing to Meijer's G-function, have been used very often in the univalent function theory, like the known operators of: Biernacki, Komatu, Libera, Rusheweyh, Owa and Srivastava, Carlson and Shaffer, Saigo, Hohlov, etc. (see the examples in [8, Ch.5], and details in Kiryakova, Saigo and Owa [10], Kiryakova, Saigo, Srivastava [11]). Thus, the results below give as corollaries corresponding properties of all these operators.

3. Distortion Inequalities in the Classes $S_\alpha(n)$ and $L_\alpha(n)$

A function $f(z)$ belonging to $S(n)$ is said to be *starlike of order* α $(0 \leqq \alpha < 1)$ if and only if it satisfies the inequality

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \qquad (z \in U) \tag{25}$$

and this subclass is denoted by $S_\alpha(n)$. Further, $f(z) \in S(n)$ is said to be *convex of order* α $(0 \leqq \alpha < 1)$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U) \tag{26}$$

and the subclass is denoted by $K_\alpha(n)$. We note that $f(z) \in K_\alpha(n)$ if and only if $zf'(z) \in S_\alpha(n)$, and also for any $0 \leqq \alpha < 1$,

$$S_\alpha(n) \subseteq S_0(n), \qquad K_\alpha(n) \subseteq K_0(n) \qquad \text{and} \qquad K_\alpha(n) \subset S_\alpha(n).$$

The classes $S_\alpha(n)$ and $K_\alpha(n)$ have been recently studied by Srivastava, Owa and Chatterjea [30]. For $n = 1$, these denotations are usually used as $S_\alpha(1) = S^*(\alpha),\ K_\alpha(1) = K(\alpha)$, which are introduced earlier by Robertson [20]. Especially, taking $\alpha = 0$, we obtain the well-known classes S^* and K of starlike and convex functions in U, respectively.

In the class $T(n)$ of functions (2) with negative coefficients, we take now the respective intersections for $0 \leqq \alpha < 1,\ n \in \mathbb{N}$:

$$T_\alpha(n) = S_\alpha(n) \cap T(n), \quad L_\alpha(n) = K_\alpha(n) \cap T(n). \tag{27}$$

The latter classes were considered by Chatterjea [1] and in particular, case $n = 1$ gives the Silverman's classes $T^*(\alpha), L(\alpha)$ [27].

For functions of these classes we propose here some *distortion inequalities* in terms of the generalized fractional calculus operators (13).

We need first the following two lemmas given by Chatterjea [1].

Lemma 1. *Let the function* $f(z) \in A(n)$. *Then* $f(z)$ *is in the class* $T_\alpha(n)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{k-\alpha}{1-\alpha}\, a_k \leqq 1. \tag{28}$$

Lemma 2. *Let the function* $f(z) \in A(n)$. *Then* $f(z)$ *is in the class* $L_\alpha(n)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{k(k-\alpha)}{1-\alpha} a_k \leqq 1. \tag{29}$$

Applying Lemma 1 and Theorem 1, we obtain

Theorem 3. *Let conditions* (12) *be satisfied and the function* $f(z) \in A(n)$ *belong to the class* $T_\alpha(n)$. *Then the following inequalities hold for* $z \in U$:

$$\left|\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)\right| \geqq |z| - \frac{1-\alpha}{n+1-\alpha}\, \theta(n+1)\, |z|^{n+1} \tag{30}$$

and

$$\left|\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)\right| \leqq |z| + \frac{1-\alpha}{n+1-\alpha}\, \theta(n+1)\, |z|^{n+1}, \tag{31}$$

where the multiplier $\theta(n+1)$ *is defined as in* (15), *namely:*

$$\theta(n+1) = \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + (n+1)/\beta_i)\,\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + (n+1)/\beta_i)\,\Gamma(\gamma_i + 1 + 1/\beta_i)} > 0. \tag{32}$$

Equalities in (30) *and* (31) *are attained by the function*

$$f(z) = z - \frac{1-\alpha}{n+1-\alpha}\, z^{n+1}. \tag{33}$$

Analogously, an application of Lemma 2 leads to

Theorem 4. *Let conditions* (12) *be satisfied and the function* $f(z) \in A(n)$ *belong to the class* $L_\alpha(n)$. *Then the following inequalities hold for* $z \in U$:

$$\left| \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) \right| \geqq |z| - \frac{1-\alpha}{n+1-\alpha}\,\frac{\theta(n+1)}{n+1}\,|z|^{n+1} \tag{34}$$

and

$$\left| \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) \right| \leqq |z| + \frac{1-\alpha}{n+1-\alpha}\,\frac{\theta(n+1)}{n+1}\,|z|^{n+1}, \tag{35}$$

where the multiplier $\theta(n+1)$ *is defined as in* (32). *Equalities in* (34) *and* (35) *are attained by the function*

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)}\, z^{n+1}. \tag{36}$$

Proof of Theorems 3 and 4. The main point in this proof is that the multiplier function $\theta(k)$ is nonicreasing for $k \geqq n+1$. To verify this, let us start from that the known Digamma–function $\psi(x) = \Gamma'(x)/\Gamma(x)$ is increasing for all $x > 0$, for which $\psi'(x) > 0$ for all $x \neq -n$. This follows from the known logarithmic convexity of the Gamma-function, for example, from

$$\psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}, \quad x \neq 0, -1, -2, \ldots$$

(see the representation for $\psi^{(n)}(x)$ in [15, App. II.3]). Then,

$$\psi(x+\varepsilon) = \frac{\Gamma'(x+\varepsilon)}{\Gamma(x+\varepsilon)} > \frac{\Gamma'(x)}{\Gamma(x)} = \psi(x) \quad \text{for} \quad \varepsilon > 0,$$

or, for the auxiliary function $\widetilde{\Gamma}(x) := \Gamma(x+\varepsilon)/\Gamma(x)$ has a positive derivative

$$\widetilde{\Gamma}'(x) = \frac{\Gamma'(x+\varepsilon)\Gamma(x) - \Gamma(x+\varepsilon)\Gamma'(x)}{\Gamma^2(x)} > 0 \quad \text{for} \quad x > 0,\ \varepsilon > 0.$$

Then, $\widetilde{\Gamma}(x)$ is also an increasing function, and so,

$$\frac{\Gamma(x+\varepsilon)}{\Gamma(x)} \geqq \frac{\Gamma(y+\varepsilon)}{\Gamma(y)} \quad \text{whenever} \quad x \geqq y > 0.$$

This, by the replacement $\varepsilon \mapsto 1/\beta_i, x \mapsto a_i + k/\beta_i, y \mapsto b_i + k/\beta_i$ (according to the notations in (17) and $a_i \geqq b_i > 0$) gives

$$\frac{\Gamma(a_i + (k+1)/\beta_i)}{\Gamma(a_i + k/\beta_i)} \geqq \frac{\Gamma(b_i + (k+1)/\beta_i)}{\Gamma(b_i + k/\beta_i)} \quad (i = 1, \ldots, m),$$

therefore the required nonicreasing property for $\theta(k)$ follows:

$$\frac{\theta(k)}{\theta(k+1)} = \prod_{i=1}^{m} \frac{\Gamma(b_i + k/\beta_i)}{\Gamma(b_i + (k+1)/\beta_i)} \cdot \frac{\Gamma(a_i + (k+1)/\beta_i)}{\Gamma(a_i + k/\beta_i)} \geqq 1. \tag{37}$$

Hence,

$$0 < \theta(k) \leqq \theta(n+1) \quad \text{for each} \quad k \geqq n+1, \tag{38}$$

and for $f(z)$ of the form (2),

$$\begin{aligned} \left|\widetilde{I}_{(\beta_i),m}^{(\gamma_i),(\delta_i)} f(z)\right| &\geqq |z| - \left| \sum_{k=n+1}^{\infty} \theta(k)\, a_k z^k \right| \\ &\geqq |z| - \theta(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \geqq |z| - \theta(n+1)|z|^{n+1} \frac{1-\alpha}{n+1-\alpha}, \end{aligned}$$

since in view of Lemma 1 (see (28)), we have also

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\alpha}{n+1-\alpha}.$$

Thus, inequality (30) is obtained. Next inequality (31) can be proved similarly and Theorem 4 follows in analogous way by the application of Lemma 2. ■

Corollary 3. *If we set $n = 1$ and $\alpha = 0$, we obtain for the subclasses of starlike and convex functions in U, respectively*

$$f \in S^* \cap T(1) \implies |\widetilde{I}f(z)| \geqq |z| - \frac{\theta(2)}{2}\, |z|^2, \quad |\widetilde{I}f(z)| \leqq |z| + \frac{\theta(2)}{2}\, |z|^2$$

$$f \in K \cap T(1) \implies |\widetilde{I}f(z)| \geqq |z| - \frac{\theta(2)}{4}\, |z|^2, \quad |\widetilde{I}f(z)| \leqq |z| + \frac{\theta(2)}{4}\, |z|^2$$

with the multiplier

$$\theta(2) = \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + 2/\beta_i)\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + 2/\beta_i)\Gamma(\gamma_i + 1 + 1/\beta_i)}.$$

Remark. The case $m = 1$ gives respective estimates for the classical Erdélyi–Kober operators (9).

As applications of the above general results, we can derive the same kind ones for the operators by Saigo ([21]–[23], [31]), and by Hohlov ([3], [4]) as well as for the fractional integrals and derivatives involving the Appell's F_3-function, recently studied by Saigo et al. [24], [25]. All these cases fall in the scheme of the G-function generalized fractional calculus operators (21) and the details are given in Kiryakova, Saigo and Owa [10].

4. Some Characterization Theorems in the Classes $S^*(n)$ and $K(n)$

Now we consider some sufficient conditions for the operators of generalized fractional calculus to produce starlike and convex functions. Namely, we denote by $S^*(n)$ the subclass of $A(n)$ of functions satisfying (25) with $\alpha = 0$, i.e. $S^*(n) := S_0(n)$. Analogously, $K(n) := K_0(n)$ is the subclass of $A(n)$ of functions $f(z)$ satisfying (26) with $\alpha = 0$.

From Silverman's results [27], one can formulate the following auxiliary lemmas.

Lemma 3. *If the function $f(z) \in A(n)$ satisfies the condition*

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq 1, \tag{39}$$

then $f(z) \in S^*(n)$. The equality in (39) is attained by the function

$$g_1(z) = z + \varepsilon(n+1) \sum_{k=n+1}^{\infty} \frac{z^k}{k^2(k+1)} \quad (\varepsilon = \text{const},\ |\varepsilon| = 1,\ z \in U). \tag{40}$$

Lemma 4. *If the function $f(z) \in A(n)$ satisfies the condition*

$$\sum_{k=n+1}^{\infty} k^2|a_k| \leqq 1, \tag{41}$$

then $f(z) \in K(n)$. The equality in (41) is attained by the function

$$g_2(z) = z + \varepsilon(n+1) \sum_{k=n+1}^{\infty} \frac{z^k}{k^3(k+1)} \quad (\varepsilon = \text{const},\ |\varepsilon| = 1,\ z \in U). \tag{42}$$

For the generalized fractional integrals (13) we obtain then the following sufficient conditions.

Theorem 5. *Under the condition (12), if the function $f(z) \in A(n)$ satisfies*

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq \frac{1}{\theta(n+1)}, \tag{43}$$

then $\widetilde{I}_{(\beta_i),m}^{(\gamma_i),(\delta_i)} f(z)$ belongs to the class $S^*(n)$.

Proof. We use again the inequality (38). Then, for the function

$$\widetilde{I}f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$$

with coefficients $b_k = \theta(k) a_k$, we obtain

$$\sum_{k=n+1}^{\infty} k b_k \leqq \theta(n+1) \sum_{k=n+1}^{\infty} k a_k \leqq 1. \qquad \blacksquare$$

Analogously, using Lemma 4, we obtain

Theorem 6. *Under the condition* (12), *if the function* $f(z) \in A(n)$ *satisfies*

$$\sum_{k=n+1}^{\infty} k^2 |a_k| \leqq \frac{1}{\theta(n+1)}, \tag{44}$$

then $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)$ *belongs to the class* $K(n)$.

Remark. Examples of functions satisfying conditions (43), (44) are the functions

$$g_3(z) = z + \frac{1}{\theta(k_0)} \frac{z^{k_0}}{k_0} \quad \text{and} \quad g_4(z) = z + \frac{1}{\theta(k_0)} \frac{z^{k_0}}{k_0^2},$$

respectively, with some $k_0 > n+1$.

5. Special cases

Obviously, putting in results here $\beta_i = 1\,(i = 1, \ldots, m)$, we obtain analogues of Theorems 1–6 for the generalized fractional integration operators with G-function kernels (see Kiryakova, Saigo and Owa [10]). Then, the same type results follow for a number of integral operators (or, integro-differential and differential operators, when considering the respective generalized fractional derivatives) that are rather popular in univalent function theory but follow as special cases (mainly for $m = 1, 2$ and one example for $m = 3$).

In Saigo [21], [23], the following operator of generalized fractional integration and differentiation that involve the *Gauss hypergeometric function* have been introduced:

$$I^{\alpha,\beta,\eta} f(z) = z^{-\alpha-\beta} \int_0^z \frac{(z-\xi)^{\alpha-1}}{\Gamma(\alpha)} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{\xi}{z}\right) f(\xi) d\xi \tag{45}$$

for real parameters $\alpha > 0, \beta, \eta$. First, operator (45) has been considered for real-valued functions and used for solving boundary value problems [22], [31] for the Euler-Darboux equation, but Srivastava, Saigo and Owa (see for example, [32], [13]) have applied it to classes of univalent functions.

The "normalized" operator of (45) falls in the scheme of operators (13) with $m = 2$, namely:

$$\widetilde{I}^{\alpha,\beta,\eta} f(z) := \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I^{\alpha,\beta,\eta} f(z) = \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} I^{(\eta-\beta,0),(-\eta,\alpha+\eta)}_{(1,1),2} f(z) \tag{46}$$

and has respectively, multiplier sequence and convolution function of the forms:

$$\theta(k) = \frac{(-\beta+\eta+2)_{k-1} k!}{(-\beta+2)_{k-1}(\alpha+\eta+2)_{k-1}},$$

$$h(z) = z + \frac{(-\beta+\eta+2)_n (n+1)!}{(-\beta+2)_n (\alpha+\eta+2)_n} z^{n+1} {}_3F_2\left(\begin{matrix} 1, -\beta+\eta+2+n, 2+n \\ -\beta+2+n, \alpha+\eta+2+n \end{matrix}; z\right).$$

Especially in the class $A = A(1)$, its convolutional representation turns into:

$$\widetilde{I}^{\alpha,\beta,\eta} f(z) = h(z) * f(z) \quad \text{with} \quad h(z) = z + z^2 \, {}_3F_2\left(\begin{matrix} 1, -\beta+\eta+2, 2 \\ -\beta+2, \alpha+\eta+2 \end{matrix}; z\right).$$

For the corresponding results in the classes we consider, for any $n \in \mathbb{N}$ under conditions $\beta - \eta < 2, \alpha + \eta \geqq 0, \eta \leqq 0$ (see Kiryakova, Saigo and Owa [10]).

In [3], [4] Hohlov introduced a generalized fractional integration operator defined by means of the Hadamard product with an arbitrary Gauss hypergeometric function:

$$\mathbf{F}(a,b,c)f(z) := z\ {}_2F_1(a,b;c;z) * f(z). \tag{47}$$

This three-parameter family of operators contains as special cases most of the known linear integral or differential operators, already used in univalent functions theory, namely: the *Biernacki operator, Rusheweyh derivative, generalized Libera operator and its inverse, Carlson-Shaffer operator*, etc. For details, see Hohlov [3], [4], Kiryakova [8], Kiryakova et al. [11], [12].

This rather general *Hohlov operator* (47) also follows as a *particular case of generalized fractional integrals* (13):

$$\mathbf{F}(a,b,c)f(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ I_{(1,1),2}^{(a-2,b-2),(1-a,c-b)}f(z) = \widetilde{I}_{(1,1),2}^{(a-2,b-2),(1-a,c-b)}f(z). \tag{48}$$

Thus, Theorems 1–6 give corresponding results for this operator, and also for all its special cases. The conditions (12) now are: $0 < a \leqq 1,\ 0 < b \leqq c$. We will refer here only to the form of its multipliers and convolution function, namely:

$$\theta(k) = \frac{(a)_{k-1}(b)_{k-1}}{(1)_{k-1}(c)_{k-1}}, \quad \text{resp.} \quad \theta(n+1) = \frac{(a)_n(b)_n}{n!(c)_n},$$

$$h(z) = z + \frac{(a)_n(b)_n}{n!(c)_n}\ z^{n+1}\ {}_3F_2\left(1, a+n, b+n; c+n, 1+n; z\right) \quad \text{in} \quad A(n),$$

resp. in the class $A = A(1):\ h(z) = z\ {}_2F_1(a,b;c;z)$, a result that conforms with original Hohlov's representation (47).

In [24], [25] Saigo and his co-worker investigated in details the operator of generalized fractional integration which involves the so-called *Appell's F_3-function*:

$$I(\alpha,\alpha',\beta,\beta';\gamma)f(z) = z^{-\alpha}\int_0^z \frac{(z-\xi)^{\gamma-1}}{\Gamma(\gamma)}\xi^{-\alpha'}\,F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{\xi}{z},1-\frac{z}{\xi}\right)f(\xi)d\xi, \tag{49}$$

but can be decomposed also as products of three Erdélyi–Kober operators (9). As shown by Kiryakova [8], this is an example of generalized fractional integrals (7), (13) of multiplicity $m = 3$, and could be represented also in the form

$$I(\alpha,\alpha',\beta,\beta';\gamma)f(z) = z^{-\alpha-\alpha'+\gamma}\int_0^1 G_{3,3}^{3,0}\left[\sigma \,\middle|\, \begin{matrix} \alpha-\alpha'+\beta, \gamma-2\alpha', \gamma-\alpha'-\beta' \\ \alpha-\alpha', \beta-\alpha', \gamma-2\alpha'-\beta' \end{matrix}\right] f(z\sigma)d\sigma.$$

Then,

$$I(\alpha,\alpha',\beta,\beta';\gamma)f(z) = z^{-\alpha-\alpha'+\gamma}\ I_{(1,1,1),3}^{(\alpha-\alpha',\beta-\alpha',\gamma-2\alpha'-\beta'),(\beta,\gamma-\alpha'-\beta,\alpha')}f(z). \tag{50}$$

and for the *"normalized" F_3-operator* of form (13):

$$\widetilde{I}f(z) = \widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z) := z^{\alpha+\alpha'-\gamma}\ I(\alpha,\alpha',\beta,\beta';\gamma)f(z)$$

we can apply *all the results for classes of univalent functions, already obtained in Theorems 1–6.* Let us mention that in this case the convolution function $h(z)$ expresses in terms of the ${}_4F_3$-function. Details can be seen in Kiryakova, Saigo and Owa [10].

Now, we consider some *two examples of integral operators*, studied recently in classes of univalent functions, *that fall essentially in the case of generalized fractional integration operators with* $(\beta_1, \beta_2, \ldots, \beta_m) \neq (1, 1, \ldots, 1)$.

These are integral operators, considered in several modified forms by Raina et al. (see Raina [16], Raina end Bolia [17], Raina, Saigo and Choi [19], Raina and Kalia [18]), and others).

The first operator, in the case of functions $f(z)$ of the class $A(n)$, is (see for example [18, p.337, (2.3), (2.5)]):

$$T_C^A(a, c; n) f(z) = \Phi_C^A(a, c; n; z) * f(z) \tag{51}$$

with

$$\begin{aligned}\Phi_C^A(a, c; n; z) &= \frac{\Gamma(c + (p-1)C)}{\Gamma(a + (p-1)A)} z^n \sum_{k=0}^{\infty} \frac{\Gamma(a + (p-1)A + nA)}{\Gamma(c + (p-1)C + nC)} z^k \\ &= \frac{\Gamma(c + (p-1)C)}{\Gamma(a + (p-1)A)} z^n \, {}_2\Psi_1 \left(\begin{matrix} (1,1), (a - A + nA, A) \\ (c - C + nC, C) \end{matrix} ; z \right),\end{aligned} \tag{52}$$

and *the second* (note that $\beta > 0$ here was denoted by m in the original papers by Raina et al.) is a composition of two operators as above, $T_C^A(a, c) := T_C^A(a, c; 1)$ ($n = 1$ is taken for simplicity):

$$\begin{aligned}M_{z;\beta}^{\lambda,\mu,\eta} f(z) :&= T_\beta^\beta(1 + \beta, 1 - \mu + \beta)\, T_\beta^\beta(1 + \eta - \mu + \beta, 1 + \eta - \lambda + \beta)\, f(z) \\ &= \frac{\Gamma(1 - \mu + \beta)\Gamma(1 + \eta - \lambda + \beta)}{\Gamma(1 + \beta)\Gamma(1 + \eta - \mu + \beta)} z^{\mu/\beta} D_{0,z;1/\beta}^{\lambda,\mu,\eta} f(z).\end{aligned} \tag{53}$$

Then, for $0 \leqq \lambda < 1;\ \mu, \eta \in \mathbb{R},\ \beta > 0, \beta > \max\{\lambda - \eta - 1, \mu - 1\}$,

$$D_{0,z;\beta}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz^\beta} \left\{ \frac{z^{-\beta(\mu-\lambda)}}{\Gamma(1-\lambda)} \int_0^z (z^\beta - t^\beta)^{-\lambda} \, {}_2F_1 \left(\mu - \lambda, 1 - \eta; 1 - \lambda; 1 - \frac{t^\beta}{z^\beta} \right) f(t) dt^\beta \right\} \tag{54}$$

is the fractional differential operator, corresponding to the so-called *modified Saigo operator* $I_{0,z;\beta}^{\lambda,\mu,\eta}$ (see the same papers by Raina et al., and compare with expressions in (45), (46)),

$$I_{0,z;\beta}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\beta(\lambda+\mu)}}{\Gamma(\lambda)} \int_0^z (z^\beta - t^\beta)^{\lambda-1} \, {}_2F_1 \left(\lambda + \mu, -\eta; \lambda; 1 - \frac{t^\beta}{z^\beta} \right) f(t) dt^\beta \tag{55}$$

in our denotations (7), (13).

It is seen then, that for $\beta = 1$, $n = 1$, $A = C = 1$, $0 < a < c$, the operator $T_C^A(a, c; n)$ reduces to the *Carlson-Shaffer integral operator* $L(a, c)$, defined by Hadamard product with a Gauss function, and easily seen to be a special case of the Erdélyi–Kober operators (8) (see e.g. [10]):

$$\begin{aligned}\mathbf{L}(a, c) f(z) &= \Phi(a, c; z) * f(z) = \{ z \, {}_2F_1(1, a; c; z) \} * f(z) \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 (1 - \sigma)^{c-a-1} \sigma^{a-2} f(z\sigma) d\sigma = \frac{\Gamma(c)}{\Gamma(a)} I_1^{a-2, c-a} f(z).\end{aligned} \tag{56}$$

The operators (53), (54), i.e. $M_{z;\beta}^{\lambda,\mu,\eta}$ or $D_{0,z;\beta}^{\lambda,\mu,\eta}$, reduce for $\beta = 1$ to the *hypergeometric fractional derivative* (resp. *integral* (45)) with a Gauss function, *studied by Saigo et al.*

The operators (51)–(52) with $A = C = 1/\beta$ and (53)–(55) are special cases of the generalized fractional integrals (7), (13), resp. for $m = 1$ and $m = 2$ (with $A_1 = C_1 = A_2 = C_2 = 1/\beta$, i.e. $\beta_1 = \beta_2 = \beta > 0$). Evidently, m-tuple compositions of operators (51), (52) give operators of form (13) in the general case $m > 1$.

Results for above two operators have been obtained by Raina et al., for example as follows: an analogue of Lemma 0 (for $D_{0,z;\beta}^{\lambda,\mu,\eta}$), and respective operational properties of both operators $D_{0,z;\beta}^{\lambda,\mu,\eta}$, $M_{z;\beta}^{\lambda,\mu,\eta}$ – in Raina [16], where as applications some inequalities for the Wright functions ${}_p\Psi_q$, (5) have been derived; some characterization theorems – for $D_{0,z;\beta}^{\lambda,\mu,\eta}$ in Raina, Saigo and Choi [19], and – for $M_{z;\beta}^{\lambda,\mu,\eta}$ in Raina and Kalia [18], etc. Evidently, the results presented here for generalized fractional calculus operators (13) give, as special cases, also a series of other corresponding analogues for the mentioned two operators.

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