On Fundamental Lemmas Developed for Stochastic Approximation Processes

Masafumi Watanabe

Department of Applied Mathematics, Fukuoka University
Fukuoka 814–0180, Japan

(Received May 31, 2003)

1. Introduction

In this paper we will prove fundamental lemmas which are used in solving various aspects of the problems stochastic approximation processes. In [1] , C.Derman and J.Sacks give the following two lemmas which are at the basis of stochastic approximation theorems.

Lemma A. *Let $\{\alpha_n\}, \{v_n\}, \{w_n\}, \{a_n\}$ and $\{x_n\}$ be sequences of real numbers such that*

(i) $\{\alpha_n\}, \{v_n\}, \{a_n\}$ *and* $\{x_n\}$ *are nonnegative,*

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} v_n < \infty$, $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} w_n$ *converges, and, for some integer N and for all $n \geq N$,*

(iii) $x_{n+1} \leq \max\{\alpha_n, (1+v_n)x_n - a_n + w_n\}$.

Then, $\lim_{n\to\infty} x_n = 0$.

Lemma B. *Let $\{\alpha_n\}, \{a_n\}$ and $\{x_n\}$ be as in Lemma A. Suppose*

(i) $\sum_{n=1}^{\infty} v_n$ *converges,* $\sum_{n=1}^{\infty} v_n^2 < \infty$,

(ii) $w_n \geq 0$ *and* $\sum_{n=1}^{\infty} w_n < \infty$.

Then, $\lim_{n\to\infty} x_n = 0$.

In [2], J.H.Venter also gives the following extended version of Lemma B.

Lemma C. *Let* $\{v_n\}, \{w_n\}$ *and* $\{a_n\}$ *be as in Lemma B. Suppose that* $\{x_n\}$ *is a nonnegative sequence such that , for some integer* N *and for all* $n \geq N$,

$$x_{n+1} \leq \max\{\alpha, (1+v_n)x_n - a_n + w_n\},$$

where $\alpha > 0$. *Then,* $\overline{\lim_{n\to\infty}} x_n \leq \alpha$.

In [3], the following extended version of Lemma A is given the case when $\lim_{n\to\infty} a_n = 0$ holds.

Lemma D. *Let* $\{v_n\}$ *be as in Lemma A. Suppose*

(i) $a_n \geq 0, \quad \lim_{n\to\infty} a_n = 0, \quad \sum_{n=1}^{\infty} a_n = \infty,$

(ii) $\lim_{n\to\infty} \max_{n \leq k < a(n,T)} |\sum_{i=n}^{k} w_i| = 0 \qquad$ *for each* $\quad T > 0$,

where $\qquad a(n,T) = \max\{k \geq n | \sum_{i=n}^{k} a_i \leq T\},$

(iii) $\{x_n\}$ *is a nonnegative sequence such that for some integer* N *and for all* $n \geq N$,

$$x_{n+1} \leq \max\{\alpha, (1+v_n)x_n - a_n + w_n\},$$

where $\alpha > 0$. *Then,* $\overline{\lim_{n\to\infty}} x_n \leq \alpha$.

The above lemmas are very useful in solving various aspects of problems of stochastic approximation method. In the paper,we willl give fundamental lemmas which are extended version of Lemmas A to D. Especialy we will inrestigate the case when $\lim_{n\to\infty} a_n = 0$ holds but "$\sum v_n$ converges" and "$\sum w_n$ converges" may not be fulfilled.

2. Fundamental lemmas

Thoughout the paper let $\{a_n\}$ be a sequence of nonnegative numbers such that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} a_n = \infty. \tag{2.1}$$

For the above sequence $\{a_n\}$ and for any $T > 0$ we define a sequence of positive integer $\{a(n, T)\}$ by

$$a(n,T) = \begin{cases} \max\{k \geq n | \sum_{i=n}^{k} a_i \leq T\} & \text{if } a_n < T \\ n & \text{if } a_n \geq T. \end{cases} \tag{2.2}$$

Then it follows that, for any $T > 0$ there exists a positive integer $N(T)$ such that, for all $n \geq N(T)$

$$n < a(n,T) \tag{2.3}$$

$$T - 1 < \sum_{i=n}^{a(n,T)} a_i \leq T \tag{2.4}$$

$$\lim_{n\to\infty} \sum_{i=n}^{a(n,T)} a_i = T. \tag{2.5}$$

Lemma 1. *Let $\{v_n\}, \{w_n\}$ be sequences of real numbers such that, for any $T > 0$,*

$$\lim_{n\to\infty} \max_{n \leq k < a(n,T)} |\sum_{i=n}^{k} v_i| = 0, \tag{2.6}$$

$$\lim_{n\to\infty} \sum_{i=n}^{a(n,T)} v_i^2 = 0, \tag{2.7}$$

$$\lim_{n\to\infty} \max_{n \leq k < a(n,T)} |\sum_{i=n}^{k} w_i \prod_{j=i+1}^{k} (1 + v_j)| = 0. \tag{2.8}$$

Suppose that $\{x_n\}$ is a sequence of nonnegative numbers such that, for some integer N and for all $n \geq N$,

$$x_{n+1} \leq \max\{\alpha, (1 + v_n)x_n - a_n + w_n\}, \tag{2.9}$$

where $\alpha > 0$. If a positive constant K^* exists with $x_n \leq K^*$ for infinitely many n, then, $\overline{\lim_{n\to\infty}} x_n \leq \alpha$.

The following Lemma 2 is an extension version of Lemmas A to D.

Lemma 2. *Let $\{v_n\}, \{w_n\}$ and $\{x_n\}$ be as in Lemma 1. If instead of (2.6) and (2.7) the following conditions hold,*

$$\sum_{n=1}^{\infty} v_n \quad \textit{converges and} \quad \sum_{n=1}^{\infty} v_n^2 < \infty, \tag{2.10}$$

then, $\overline{\lim\limits_{n\to\infty}}\, x_n \le \alpha$.

Lemma 3. *Let $\{v_n\}$ be sequence of real numbers which satsfies (2.6) and $\{w_n\}$ be as in Lemma 1. If $\{x_n\}$ is a sequence of nonnegative numbers such that, for some integer N and for all $n \ge N$,*

$$x_{n+1} \le \max\{\alpha, (1 - a_n + v_n)x_n + w_n\}, \tag{2.11}$$

where $\alpha > 0$, then, $\overline{\lim\limits_{n\to\infty}}\, x_n \le \alpha$.

Remark 1. Let $\{v_n\}$ and $\{w_n\}$ be sequences of real numbers. If $\{v_n\}$ satisfies the following conditions,

$$\sup_n \sum_{i=n}^{a(n,T)} |v_i| < \infty \qquad \text{for each} \quad T > 0 \tag{2.12}$$

then (2.8) is equivalent to the following condition,

$$\lim_{n\to\infty} \max_{n \le k < a(n,T)} \Big|\sum_{i=n}^{k} w_i\Big| = 0 \qquad \text{for each} \quad T > 0. \tag{2.13}$$

Remark 2. Let us set $a_n = n^{-1}, u_n^{(1)} = (\log(n+1))^{-1}$ and $u_n^{(2)} = 1$ if $n = m^2, u_n^{(2)} = 0$ if $n \ne m^2$ for $n \ge 1$. Then it holds that $\lim\limits_{n\to\infty} \sum\limits_{i=n}^{a(n,T)} a_i u_i = 0$ for each $T > 0$, where $u_n = n_n^{(1)} + u_n^{(2)}$. But the sequence $\{u_n\}$ does not converges and $\sum\limits_{n=1}^{\infty} a_n u_n = \infty$.

Remark 3. Let us set $a_n = n^{-1}, v_n = (-1)^{n+1}n^{-1}$ and $w_n = (n\log(n+1))^{-1}$ (or $(-1)^{n+1}n^{-1}$) for $n \ge 1$. In this case we can not apply Lemmas A to D but the conditions of $\{v_n\}$ and $\{w_n\}$ in Lemma 2 are fulfilled.

Remark 4. Let us set $a_n = n^{-1}$ and $v_n = w_n = a_n u_n$ for $n \geq 1$, where u_n is as in Remark 2. In this case we can not apply Lemmas A to D and Lemma 2. But (2.6) and (2.8) are fulfilled. Thus, (2.11) is stronger than (2.9) but the conditons of $\{v_n\}$ are more general than the conditions in Lemmas A to D and Lemma 2. An application of Lemma 3 will be given in Section 4.

3. Proof of Lemmas

The following lemma can be easily proved.

Lemma 3.0. *Let $\{\xi_n\}$ be a sequence of nonnegative numbers. And let a, b, c and t be positive numbers.*

(i) *If $\{\xi_n\}$ satisfies $\xi_{n+1} \leq \max\{a, (1+b)\xi_n - t\} + c$ for $n \geq 0$ and if $\max\{(1+b)\xi_0 - a, ab + (1+b)c\} \leq t$ holds then, $\sup_{1 \leq n} \xi_n \leq a + c$.*

(ii) *If $\{\xi_n\}$ satisfies $\xi_{n+1} \leq \max\{a, \xi_n - t\} + c$ for $n \geq 0$ and if $c \leq t$ holds then, $\sup_{1 \leq n} \xi_n \leq \max\{a, b + c, \xi_0\}$.*

(iii) *If $\{\xi_n\}$ satisfies $\xi_{n+1} \leq \max\{a, b\,\xi_n\} + c$ for $n \geq 0$ and if $0 < b < 1$ holds then, $\sup_{1 \leq n} \xi_n \leq \max\{a, b\,\xi_0\} + c(1-b)^{-1}$.*

Proof of Lemma 1. Let us define

$$V(i,k) = \prod_{j=i}^{k}(1+v_j) \quad \text{if} \quad k \geq i \;, \; V(k+1,k) = 1,$$

$$W(i,k) = \sum_{j=i}^{k} w_j V(j+1,k) \quad \text{if} \quad k \geq i \;, \; W(k+1,k) = 0,$$

Lemma 3.1. *Let $\{v_n\}, \{w_k\}$ be as in Lemma 1. Then it holds that, for any $T > 0$,*

$$\lim_{n\to\infty} \max_{n \leq i \leq k < a(n,T)} V(i,k) = 1. \tag{3.1}$$

$$\lim_{n\to\infty} \min_{n \leq i \leq k < a(n,T)} V(i,k) = 1. \tag{3.2}$$

Proof. Take $T > 0$ and fix. (2.6) implies $\lim_{n\to\infty} v_n = 0$. There is no loss of generality in supposing that, for all $n \geq N$,

$$|v_n| \leq \frac{1}{\sqrt{2}}. \tag{3.3}$$

Using the inequality $\log(1+x) \leq x$ which holds for $x > -1$, we get

$$V(i,k) \leq \exp\left(2 \max_{n\leq k<a(n,T)} \left|\sum_{i=n}^{k} v_i\right|\right) \quad \text{for } N \leq n \leq i \leq k < a(n,T).$$

Hence we have from (2.6)

$$\overline{\lim_{n\to\infty}} \max_{n\leq i\leq k<a(n,T)} V(i,k) \leq 1. \tag{3.4}$$

And we also have

$$\overline{\lim_{n\to\infty}} \max_{n\leq i\leq k<a(n,T)} \prod_{j=i}^{k}(1-v_j) \leq 1. \tag{3.5}$$

Using the inequality $\log(1-x) \geq (-x)(1-x)^{-1}$ which holds for $0 \leq x < 1$, we get from (3.3) $\prod_{j=i}^{k}(1-v_j^2) \geq \exp(-2\sum_{j=n}^{k} v_j^2)$ for $N \leq n \leq i \leq k < a(n,T)$.
Hence we have from (2.7)

$$\underline{\lim_{n\to\infty}} \min_{n\leq i\leq k<a(n,T)} \prod_{j=i}^{k}(1-v_j^2) \leq 1. \tag{3.6}$$

Note that, for $N \leq n \leq i \leq k < a(n,T)$,

$$V(i,k) = \prod_{j=i}^{k}(1-v_j^2)\Big[\prod_{j=i}^{k}(1-v_j)\Big]^{-1} \geq \min_{n\leq i\leq k<a(n,T)} \prod_{j=i}^{k}(1-v_j^2)\Big[\max_{n\leq i\leq k<a(n,T)} \prod_{j=i}^{k}(1-v_j)\Big]^{-1}.$$

Then (3.5) and (3.6) together imply

$$\underline{\lim_{n\to\infty}} \min_{n\leq i\leq k<a(n,T)} V(i,k) \geq 1. \tag{3.7}$$

Hence, (3.4) and (3.7) together imply (3.1) and (3.2).

Using Lemma 3.1 and Lemma 3.0(i) we will prove Lemma 1.
Let ε be an arbitrary real number satisfying $0 < \varepsilon \leq 1/2$. And let us take T satisfying

$$\max\{3K^*, \frac{3}{2}(\alpha+1)\} \leq T \tag{3.8}$$

It follows from (2.3), (2.4), (2.8) and Lemma 3.1 that there exists an integer $\overline{N} = \overline{N}(\varepsilon, T) \geq N$ such that, for all $n \geq \overline{N}$,

$$n < a(n,T), \tag{3.9}$$

$$T - 1 < \sum_{i=n}^{a(n,T)-1} a_i, \tag{3.10}$$

$$\max_{n \leq i \leq k < a(n,T)} V(i,k) < 1 + \varepsilon, \tag{3.11}$$

$$\min_{n \leq i \leq k < a(n,T)} V(i,k) > 1 - \varepsilon, \tag{3.12}$$

$$\max_{n \leq k < a(n,T)} |W(n,k)| < \varepsilon/2. \tag{3.13}$$

Take $k > n \geq \overline{N}$ and iterate (2.9) back to n. This yields

$$\begin{aligned} x_k \leq \max\{ &\max_{n \leq i < k} \{\alpha V(i+1, k-1) + |W(i+1, k-1)|\}, \\ &V(n, k-1) x_n - \sum_{i=n}^{k-1} a_i V(i+1, k-1) + |W(n, k-1)|\}. \end{aligned} \tag{3.14}$$

Since $|W(i+1, k-1)| \leq |W(n, k-1)| + |W(n, i)|$, it follows that

$$\begin{aligned} x_k \leq &\max\{\alpha \max_{n \leq i < k} V(i+1, k-1),\ V(n, k-1) x_n - \sum_{i=n}^{k-1} a_i V(i+1, k-1)\} \\ &+ 2 \max_{n \leq i < k} |W(n, i)| \qquad \text{for} \quad k > n \geq \overline{N}. \end{aligned} \tag{3.15}$$

From the assumption of Lemma 1, we can choose a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $n_k \uparrow \infty$ as $n \to \infty$ and $\sup_k x_{n_k} \leq K^*$. Let us defile $\overline{k} = \min\{k \geq 1 | \overline{N} \leq n_k\}$. Now we will define a subsequence $\{\overline{n}_r\} \subset \{n\}$ by the following way. Take $\overline{n}_0 = n_{\overline{k}}$ and define $\{\overline{n}_r\}$ by $\overline{n}_r = a(\overline{n}_{r-1}, T)$ for $r = 1, 2, \cdots$. Then (3.9) implies that $\overline{n}_r \uparrow \infty$ as $r \to \infty$. And it follws from $0 < \varepsilon \leq 1/2$, (3.15) and (3.10) to (3.13) that, for $r = 0, 1, 2, \cdots$,

$$x_{\overline{n}_{r+1}} \leq \max\{(1+\varepsilon)\alpha,\ (1+\varepsilon) x_{\overline{n}_r} - \frac{T-1}{2}\} + \varepsilon.$$

Hence, using Lemma (3.0)(i) and from (3.8) we have

$$x_{\overline{n}_r} \leq (1+\varepsilon)\alpha + \varepsilon \qquad \text{for} \quad r = 1, 2, \cdots . \tag{3.16}$$

Since $\overline{n}_r \uparrow \infty$, for any $n > \overline{n}_0$ there exists an integer $r \geq 0$ such that $\overline{n}_r < n \leq \overline{n}_{r+1}$. Then, substituting (3.16) and (3.11) to (3.13) into (3.15) we have $x_n \leq (1+\varepsilon)^2 \alpha + \varepsilon(1+\varepsilon) + \varepsilon$

for $n > \bar{n}_0$. Hence we have $\overline{\lim_{n\to\infty}} x_n \le \alpha$. Thus the proof is completed.

Proof of Lemma 2.

Lemma 3.2. *Let* $\{v_n\}, \{w_n\}$ *be as in Lemma 2. Then it holds that*

$$\lim_{n\to\infty} \prod_{n=i}^{\infty}(1+v_i) = 1, \tag{3.17}$$

$$\lim_{n\to\infty} \max_{n\le k<a(n,T)} |\sum_{i=n}^{k} w_i \sum_{j=i+1}^{\infty}(1+v_j)| = 0 \qquad \textit{for each} \quad T>0. \tag{3.18}$$

Proof. There is no loss of generality in supposing that (3.3) holds. Let us set $V_n = \prod_{i=N}^{n}(1+v_i)$ for $n \ge N$. Using the inequality $\log(1+x) \le x \ \ (x > -1)$ we get

$$0 < V_n \le \exp(|\sum_{i=N}^{n} v_i|) \qquad \text{for} \quad n \ge N. \tag{3.19}$$

Hence, (2.10) implies that the sequence $\{V_n\}$ is bounded. Now we will prove

$$\lim_{m,n\to\infty} |V_m - V_n| = 0. \tag{3.20}$$

Let $N \le m < n$. Then we have

$$|V_m - V_n| \le (\sup_m V_m)| \prod_{i=m+1}^{n} (1+v_i) - 1|. \tag{3.21}$$

By the same arguments of (3.19) we also have, for $N \le m < n$.
$0 < \prod_{i=m}^{n}(1+v_i) \le \exp(\sum_{i=m}^{n}|v_i|)$ and $0 < \prod_{i=m}^{n}(1-v_i) \le \exp(\sum_{i=m}^{n}|v_i|)$. Using the inequality $\log(1-x) \ge (-x)(1-x)^{-1}$ (for $0 \le x < 1$), we get from (3.3), for $N \le m < n$, $\prod_{i=m}^{n}(1-v_i^2) \ge \exp(-2\sum_{i=m}^{n} v_i^2)$. Hence, we have, for $N \le m < n$,

$$\exp(-2\sum_{i=m}^{n} v_i^2 - |\sum_{i=m}^{n} v_i|) \le \prod_{i=m}^{n}(1+v_i) \le \exp(|\sum_{i=m}^{n} v_i|).$$

Then (2.10) yields

$$\lim_{n,m\to\infty}\prod_{i=m}^{n}(1+v_i)=1. \tag{3.22}$$

And (3.21) and (3.22) together imply (3.20) which yields (3.17).

The equality $|\sum_{i=n}^{k} w_i \prod_{j=i+1}^{\infty}(1+v_j)| = |W(n,k)| \prod_{j=k+1}^{\infty}(1+v_j)$ and (2.8) and (3.17) together imply (3.18). Thus, the proof of the lemma is completed.

Now we will prove Lemma 2 by using Lemma 1, Lemma 3.2 and Lemma 3.0(ii). By Lemma 1 it is only enough to prove that there exists a bounded subsequence $\{x_{n_r}\} \subset \{x_n\}$ where $n_r \uparrow \infty$ as $r \to \infty$. Let us set $\widetilde{x}_n = x_n \widetilde{v}_n$ for $n \geq N$, where $\widetilde{v}_n = \prod_{i=n}^{\infty}(1+v_i)$. Then (3.14) yields, for $k > n \geq N$,

$$\widetilde{x}_k \leq \max\{\alpha \sup_{n\leq i} \widetilde{v}_i,\ \widetilde{x}_n - \sum_{i=n}^{k-1} a_i \widetilde{v}_{i+1}\} + 2 \max_{n\leq i<k} |\widetilde{W}(n,i)|, \tag{3.23}$$

where $\widetilde{W}(n,i) = W(n,i)\widetilde{v}_{i+1}$. Let us take $T \geq 3$. Then it follows from (2.3), (2.4) and Lemma 3.2 that there exists a positive integer $\widetilde{N} = \widetilde{N}(T) \geq N$ such that, for all $n \geq \widetilde{N}$,

$$n < a(n,T), \tag{3.24}$$

$$T-1 < \sum_{i=n}^{a(n,T)-1} a_i, \tag{3.25}$$

$$\frac{1}{2} \leq \widetilde{v}_n \leq \frac{3}{2}, \tag{3.26}$$

$$\max_{n\leq k<a(n,T)} |\widetilde{W}(n,k)| \leq 1/2. \tag{3.27}$$

Now we will define a subsequence $\{n_r\} \subset \{n\}$ by the following way. Take $n_0 = \widetilde{N}$ and define $\{n_r\}$ by $n_r = a(n_{r-1},T)$ for $r = 1,2,\cdots$. (3.24) implies $n_r \uparrow \infty$ as $r \to \infty$. Substituting (3.25) to (3.27) into (3.23) we have $\widetilde{x}_{n_{r+1}} \leq \max\{\frac{3}{2}\alpha, \widetilde{x}_{n_r} - 1\} + 1$ for $r = 0,1,2,\cdots$. Hence, applying Lemma 3.0(ii) we have $\widetilde{x}_{n_r} \leq \max\{\frac{3}{2}\alpha + 1, \widetilde{x}_{n_0}\}$ for $r = 1,2,\cdots$. Then we can conclude that the sequence $\{x_{n_r}\}$ is bounded. Thus the proof is completed.

Proof of Lemma 3. Let us define

$$\widehat{V}(i,k)=\prod_{j=i}^{k}(1-a_j+v_j) \quad \text{if} \quad k\ge i, \quad \widehat{V}(k+1,k)=1,$$

$$\widehat{W}(i,k)=\sum_{j=i}^{k}w_j\widehat{V}(j+1,k) \quad \text{if} \quad k\ge i, \quad \widehat{W}(k+1,k)=0.$$

And let $W(i,k)$ be defined as in the proof of Lemma 1. There is no loss of generality in supposing that $|v_n-a_n|<1$ for $n\ge N$.

Lemma 3.3. *Let $\{v_n\},\{w_n\}$ be as in Lemma 3. Then it holds that, for each $T>0$*

$$\overline{\lim_{n\to\infty}}\,\widehat{V}(n,a(n,T)-1)\le \exp(-T), \tag{3.28}$$

$$\overline{\lim_{n\to\infty}}\max_{n\le i\le k<a(n,T)}\widehat{V}(i,k)\le 1, \tag{3.29}$$

$$\lim_{n\to\infty}\max_{n\le k<a(n,T)}|\widehat{W}(n,k)|=0. \tag{3.30}$$

Proof. Using the inequality $\log(1+x)\le x$ $(x>-1)$, we get

$$\widehat{V}(n,a(n,T)-1)\le \exp\{-\sum_{i=n}^{a(n,T)-1}a_i+|\sum_{i=n}^{a(n,T)-1}v_i|\}.$$

Hence (3.28) follows from (2.5) and (2.6). The inequality $1-a_n+v_n\le 1+v_n$ and (3.4) together imply (3.29). Since $w_i=W(n,i)-(1+v_i)W(n,i-1)$, $\widehat{W}(n,k)$ can be rewritten the following formula, $\widehat{W}(n,k)=W(n,k)-\sum_{i=n}^{k}a_iW(n,i-1)\widehat{V}(i+1,k)$. Hence, (2.4), (2.8) and (3.29) together imply (3.30). Thus the proof of Lemma 3.3 is completed.

Using Lemma 3.3 and Lemma 3.0(iii) we will prove Lemma 3. First, we will prove that the sequence $\{x_n\}$ is bounded. Let us take $T\ge\log 4$. Then, it follows from (2.3) and Lemma 3.3 that there exists an integer $\widehat{N}=\widehat{N}(T)\ge N$ such that, for all $n\ge\widehat{N}$,

$$n<a(n,T) \tag{3.31}$$

$$\widehat{V}(n,a(n,T)-1)\le\exp(-T)+\frac{1}{4}\le\frac{1}{2}, \tag{3.32}$$

$$\max_{n\le i\le k<a(n,T)}\widehat{V}(i,k)\le 3/2, \tag{3.33}$$

$$\max_{n\le k<a(n,T)}|\widehat{W}(n,k)|\le 1/2. \tag{3.34}$$

Let us take $n_0 = \widehat{N}$ and define $\{n_r\}$ by $n_r = a(n_{r-1}, T)$ for $r = 1, 2, \cdots$. It follows from (3.31) that $n_r \uparrow \infty$ as $r \to \infty$. By the similar arguments in the proof of Lemma 1 we have from (2.12) and (3.32) to (3.34), $x_{n_{r+1}} \leq \max\{\frac{3}{2}\alpha, \frac{1}{2}x_{n_r}\} + 1$ for $r = 0, 1, 2, \cdots$. Hence, applying Lemma 3.0(iii) we can conclude that $\{x_{n_r}\}$ is bounded.
For any $n > n_0 = \widehat{N}$, there exists an integer $r \geq 0$ such that $n_r < n \leq n_{r+1}$. Iterating (2.12) back to n_r and substituting (3.32) to (3.34) into it, we have $x_n \leq \max\{\frac{3}{2}\alpha, \frac{1}{2}x_{n_r}\} + 1$. Hence we can conclude that $\{x_n\}$ is bounded. Let ε be an arbitrary positive number. And let us take $T_0 > \log(K_0/\alpha)$ where $K_0 > \sup_n x_n$. Then it follows from (2.3) and Lemma 3.3 that there exists an integer $N_0 = N_0(\varepsilon, T_0) \geq N$ such that , for all $n \geq N_0$,

(3.35) $$n < a(n, T_0),$$

(3.36) $$\widehat{V}(n, a(n, T_0) - 1) \leq \exp(-T_0) + \alpha\varepsilon K_0^{-1} \leq \alpha(1 + \varepsilon)K_0^{-1},$$

(3.37) $$\max_{n \leq i \leq k < a(n, T_0)} \widehat{V}(i, k) \leq 1 + \varepsilon,$$

(3.38) $$\max_{n \leq k < a(n, T_0)} |\widehat{W}(n, k)| \leq \varepsilon/2.$$

Let us take $n_0 = N_0$ and define $\{n_r\}$ by $n_r = a(n_{r-1}, T_0)$ for $r = 1, 2, \cdots$.
It follows from (3.35) that $n_r \uparrow \infty$ as $r \to \infty$. Then, for any $n > N_0$ there exists an integer $r \geq 0$ such that $n_r < n \leq n_{r+1}$. And iterating (2.12) back to n_r and substituting (3.36) to (3.38) into it we have $x_n \leq \max\{\alpha(1+\varepsilon), \alpha(1+\varepsilon)K_0^{-1}x_{n_r}\} + \varepsilon$. Since, $x_{n_r} < K_0$ for $r \geq 1$ and ε is an arbitrary positive number it follows that $\overline{\lim_{n\to\infty}} x_n \leq \alpha$. Thus the proof of Lemma 3 is completed.

4. An application of Lemma 3

In this section we will give an application of Lemma 3. We assume throughout this section that H is a real separable Hibert space with inner product $< \cdot, \cdot >$ and norm $\| \cdot \|$. Let $\boldsymbol{B}$ be the Borel field of H. Let $(\Omega, \boldsymbol{F}, P)$ be a probability space. And let $\{\boldsymbol{F}_n\}$ be increasing sequence of sub-σ-fields of $\boldsymbol{F}$. Let $M_n(x)$ be Borel measurable mappings from H into itself . For each $n \geq 1$, let a mapping $Z_n(x, \omega)$ (denoted by $Z_n(x)$) which is define for $(x, \omega) \in H \times \Omega$ and takes value in H be measurable with respect to $\boldsymbol{B} \times \boldsymbol{F}_{n+1}$.

Then we condider the following RM process.

(4.1) $$X_{n+1} = X_n - a_n\{M_n(X_n) + Z_n(X_n)\} \quad \text{for } n = 1, 2, \cdots ,$$

where X_1 is a constant element in H and $\{a_n\}$ is a sequence of nonnegative numbers.

We will make the following assumptions.

A1. $\lim_{n\to\infty} a_n = 0, \quad \sum_{n=1}^{\infty} a_n = \infty$ and let $a(n,T)$ be as in Section 2.

A2. There exists a Borel measurable mapping $M(x)$ from H into itself satisfying

(i) $\| M(x) \| \leq K(\| x \| + 1)$ for $x \in H$,

where $K > 0$, for any $\varepsilon > 0$ there exists a real number $\lambda = \lambda(\varepsilon) > 0$ such that

(ii) $\inf_{\varepsilon < \|x\|^2} (1 + \| x \|^2)^{-1} < x, M(x) > \ \geq \lambda$

and there exists a sequence of nonnegative numbers $\{\beta_n\}$ such that

(iii) $\| M_n(x) - M(x) \| \leq \beta_n(\| x \| + 1)$ for $x \in H$ and $n \geq 1$,

(iv) $\lim_{n\to\infty} \sum_{i=n}^{a(n,T)} a_i \beta_i = 0$ for each $T > 0$.

A3. There exist sequences of nonnegative numbers $\{\gamma_n\}$ and $\{\delta_n\}$ such that

(i) $\| E[Z_n(X_n) | \boldsymbol{F}_n] \| \leq \gamma_n(\| X_n \| + 1)$ $a.s.$ for $n \geq 1$,

(ii) $\lim_{n\to\infty} \sum_{i=n}^{a(n,T)} a_i \gamma_i = 0$ for each $T > 0$,

(iii) $E[\| Z_n(X_n) \|^2 | \boldsymbol{F}_n] \leq \delta_n(\| X_n \|^2 + 1)$ $a.s.$ for $n \geq 1$,

(iv) $\lim_{n\to\infty} \sum_{i=n}^{a(n,T)} a_i^2 \delta_i = 0$ for each $T > 0$.

Theorem. *Let $\{X_n\}$ be the RM process which is defined by* (4.1). *Suppose that the assumptions* A1 *to* A3 *hold. Then it follows that*

$$\lim_{n\to\infty} E \| X_n \|^2 = 0.$$

Proof. Let us define, for $x \in H$ and $n \geq 1$, $T_n(x) = x - a_n M_n(x)$. A2(i) implies

(4.2) $$\| M_n(x) \|^2 \leq 4(K^2 + \beta_n^2)(\| x \|^2 + 1) \quad \text{for} \quad x \in H \text{ and } n \geq 1.$$

Let ε be an arbitrary real number satisfying $0 < \varepsilon < 1/9$. $\lim_{n\to\infty} a_n = 0$ and A2(iv) together imply that there exists a positive integer N such that, for $n \geq N$,

(4.3) $$8a_n^2(K^2 + \beta_n^2) < \varepsilon.$$

If $\| x \|^2 \leq \varepsilon$ then we have from (4.2) and (4.3)

(4.4) $$\| T_n(x) \|^2 \leq 4\varepsilon \quad \text{for} \quad x \in H \text{ and } n \geq 1.$$

And A2(iii) implies

$$(4.5) \qquad 2| < x, M(x) - M_n(x) > | \leq 3\beta_n(\| x \|^2 + 1) \qquad \text{for } x \in H \text{ and } x \geq 1.$$

$\lim_{n\to\infty} a_n = 0$ and A2(iv) together imply that there exists a positive integer $N_1 \geq N$ such that, for $n \geq N_1$,

$$(4.6) \qquad a_n \leq \min\{(2\lambda)^{-1}, \lambda(4K^2)^{-1}\} \qquad \text{and} \quad a_n\beta_n < 1/4.$$

If $\| x \|^2 > \varepsilon$ then we have from A2(ii), (4.2), (4.5) and (4.6)

$$\begin{aligned}(4.7) \quad \| T_n(x) \|^2 &= \| x \|^2 - 2a_n(1 + \| x \|^2)^{-1} \| x \|^2 < x, M(x) > \\ &\quad - 2a_n(1 + \| x \|^2)^{-1} < x, M(x) > + 2a_n < x, M(x) - M_n(x) > \\ &\quad + a_n^2 \| M_n(x) \|^2 \\ &\leq (1 - \lambda a_n + 4a_n\beta_n) \| x \|^2 + 4a_n\beta_n \qquad \text{for } x \in H \text{ and } n \geq N_1.\end{aligned}$$

Hence, (4.4) and (4.7) together imply

$$(4.8) \qquad \| T_n(x) \|^2 \leq \max\{4\varepsilon, (1 - \lambda a_n + 4a_n\beta_n) \| x \|^2 + 4a_n\beta_n\} \qquad \text{for } x \in H \text{ and } n \geq N_1.$$

Note that A3(iii) implies $E \| Z_1(X_1) \|^2 < \infty$. Hence, applying induction on n it follows from (4.1) and A3(iii) that, for all $n \geq 1$,

$$(4.9) \qquad E \| X_n \|^2 < \infty.$$

We will write $A^+ = \max\{0, A\}$ for any real number A. And we note that $\| x \| \leq (\| x \| - \alpha)^+ + \alpha$ where $\alpha > 0$. Let us set $\alpha = 3\varepsilon^{1/2}$. Now we will show that

$$(4.10) \qquad \lim_{n\to\infty} EY_n^2 = 0,$$

where $Y_n = (\| X_n \| - \alpha)^+$. Since ε is an arbitrary positive number, it is easily seen that (4.9) and (4.10) together imply $\lim_{n\to\infty} E \| X_n \|^2 = 0$.

Let us define

$$\overline{T}_n = \begin{cases} T_n(X_n) & \text{if } \| T_n(X_n) \| \leq \alpha \\ \alpha \| T_n(X_n) \|^{-1} T_n(X_n) & \text{if } \| T_n(X_n) \| > \alpha \end{cases}$$

and $W_n = -a_n Z_n(X_n)$ for $n \geq 1$. Throught the proof let us write $T_n = T_n(X_n)$. Then we have

$$(4.11) \qquad \| T_n - \overline{T}_n \| = (\| T_n \| - \alpha)^+ \qquad \text{for } n \geq 1$$

and $\| \overline{T}_n \| \leq \alpha$ for $n \geq 1$. Hence, (4.1) yields $\| X_{n+1} \| - \alpha \leq \| X_{n+1} \| - \| \overline{T}_n \| \leq \| T_n - \overline{T}_n + W_n \|$. This yields $(\| X_{n+1} \| - \alpha)^+ \leq \| T_n - \overline{T}_n + W_n \|$ for $n \geq 1$. Hence, (4.1) and (4.11) together imply

$$(4.12) \qquad Y_{n+1}^2 \leq [(\| T_n \| - \alpha)^+]^2 + 2 < T_n - \overline{T}_n, W_n > + \| W_n \|^2 \qquad \text{for } n \geq 1.$$

Using inequalities $(A+B)^{\frac{1}{2}} \leq A^{\frac{1}{2}}(1 + \frac{B}{2A})$ (for $A > 0, B > 0$), $(1 - A + B)^{1/2} \leq 1 - \frac{A}{2} + B$ (for $0 < A < 1, B > 0$), $\| x \| \leq (\| x \| - \alpha)^+ + \alpha$, $1/2 < 1 - \lambda a_n + 4a_n\beta_n$ (for $n \geq N_1$) and $\alpha = 3\varepsilon^{1/2} < 1$, it follows that

$$(4.13) \qquad [(1 - \lambda a_n + 4a_n\beta_n) \| x \|^2 + 4a_n\beta_n]^{1/2}$$
$$\leq (1 - \frac{\lambda}{2}a_n + K_1a_n\beta_n)(\| x \| - \alpha)^+ + K_1a_n\beta_n + \alpha \qquad \text{for } x \in H \text{ and } n \geq N_1,$$

where K_1 is a positive number which is independent of n. Hence, (4.8) and (4.13) together imply

$$(4.14) \qquad \| T_n(x) \| \leq \max\{2\varepsilon^{1/2}, (1 - \frac{\lambda}{2}a_n + K_1a_n\beta_n)(\| x \| - \alpha)^+ + K_1a_n\beta_n + \alpha\}$$
$$\text{for } x \in H \text{ and } n \geq N_1.$$

Then $\alpha > 2\varepsilon^{1/2}$ and (4.14) together imply

$$(4.15) \quad (\| T_n \| - \alpha)^+ \leq (1 - \frac{\lambda}{2}a_n + K_1a_n\beta_n)(\| X_n \| - \alpha)^+ + K_1a_n\beta_n \qquad \text{for } n \geq N_1.$$

Using the inequality $(A+B)^2 \leq (1+B)A^2 + (1+B)B$ (for $B > 0$), (4.15) yields

$$(4.16) \qquad [(\| T_n \| - \alpha)^+]^2 \leq (1 - \frac{\lambda}{2}a_n + K_2a_n\beta_n)Y_n^2 + K_2a_n\beta_n \qquad \text{for } n \geq N_1,$$

where K_2 is a positive number which is independent of n. Substituting (4.16) into (4.12), noting (4.9) and taking conditional expectations for given $\boldsymbol{F}_n$, we have

$$(4.17) \quad E[Y_{n+1}^2 | \boldsymbol{F}_n] \leq (1 - \frac{\lambda}{2}a_n + K_2a_n\beta_n)Y_n^2 + K_2a_n\beta_n$$
$$+ 2 < T_n - \overline{T}_n, E[W_n | \boldsymbol{F}_n] > + E[\| W_n \|^2 | \boldsymbol{F}_n] \quad a.s. \quad \text{for } n \geq N_1.$$

Since $a_n\beta_n < 1/4$ $(n \geq N_1)$, we have from (4.15)

$$(4.18) \qquad (\| T_n \| - \alpha)^+ \leq (1 + K_1)Y_n + K_1 \qquad \text{for } n \geq N_1.$$

Hence, $\alpha < 1$, A3(i), (4.11) and (4.18) together imply

$$(4.19) \qquad 2| < T_n - \overline{T}_n, E[W_n|\boldsymbol{F}_n] > | \leq K_3 a_n \gamma_n (Y_n^2 + 1) \quad a.s. \quad \text{for } n \geq N_1,$$

where K_3 is a positive number which is independent of n. And A3(iii) implies

$$(4.20) \qquad E[\| W_n \|^2 \,|\boldsymbol{F}_n] \leq K_4 a_n^2 \delta_n (Y_n^2 + 1) \quad a.s. \quad \text{for } n \geq N_1,$$

where K_4 is a positive number which is independent of n. Substituting (4.19) and (4.20) into (4.17), noting (4.9) and taking expectations, we have

$$(4.21) \qquad E[Y_{n+1}^2] \leq (1 - \frac{\lambda}{2} a_n + v_n) E[Y_n^2] + w_n \qquad \text{for } n \geq N_1,$$

where $v_n = w_n = K_2 a_n \beta_n + K_3 a_n \gamma_n + K_4 a_n^2 \delta_n$. Then it follows from A2(iv), A3(ii) and A3(iv) that

$$\lim_{n\to\infty} \sum_{i=n}^{a(n,T)} v_i = \lim_{n\to\infty} \sum_{i=n}^{a(n,T)} w_i = 0 \qquad \text{for each } T > 0.$$

Hence, Lemma 3 and Remark 1 in Section 2 yield (4.10).

Remark. The condition A2(ii) is stronger than most RM processes. But the conditions of $\{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are weaker than them. In most RM processes, it is assumed that

$$(4.22) \qquad \sum_{n=1}^{\infty} a_n \beta_n < \infty \quad , \quad \sum_{n=1}^{\infty} a_n \gamma_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^2 \delta_n < \infty.$$

In our situations, the above condition (4.22) may not be fulfilled. An example which is not fulfilled (4.22) is given in Remark 2 in Section 2. Using Lemma 3, the *a.s.* convergence of (4.1) will be shown. In the forthcoming paper, the author will inverstigate it.

References

[1] Derman, C. and Sacks, J : On Dvoretzky's stochastic approximation theorem. *Ann. Math. Statist.*, **30**(1957), 601–605.

[2] Venter, J.H. : On Dvoretzky stochastic approximation theorems. *Ann. Math. Statist.*, **37**(1966), 1534–1544.

[3] Watanabe, M. : Almost sure convergence of RM type stochastic approximation algorithm with the boundedness assumption. *The Bull. of Cent. Res. Inst. Fukuoka Univ.*, **Vol.1 163**(1994), 121–136.