Nonlocal Problem for the Hyperbolic Equation with Fractional Derivatives in the Boundary Condition

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1. Introduction

It is well known the wide use of the classical Riemann–Liouville fractional integrals and derivatives for solutions of differential equations of hyperbolic and mixed type, for which see e.g. [5], [6], [7], [17, §40–41]. Recently it was published a series of papers [3], [4], [8], [10]–[12], devoted to the investigation of boundary problems for degenerate equations of hyperbolic type, in which the boundary conditions contain the operators of generalized fractional integration and differentiation with the Gauss hypergeometric function in the kernel, introduced by one of the authors in [9].

The present paper continues these investigations. We consider the degenerate hyperbolic equation

$$y^2 U_{xx} - U_{yy} + U_x = 0, \tag{1}$$

in a finite domain D, surrounded by the interval $(0,1)$ of the x-axis and by the characteristics

$$AC = \left\{(x,y) : x - \frac{y^2}{2} = 0,\ y \leqq 0\right\}, \quad BC = \left\{(x,y) : x + \frac{y^2}{2} = 1,\ y \leqq 0\right\} \tag{2}$$

of the equation (1). For the equation (1) we study the problem in which one boundary condition is given on $y = 0$, and the second one contains a linear combination of the classical and generalized fractional derivatives being taken at the points $\Theta_0(x) = (x/2, -\sqrt{x})$ and $\Theta_1(x) = ([x+1]/2, -\sqrt{1-x})$ of the intersection of characteristics of the equation (1) starting at the point $x \in (0,1)$ with the characteristics AC and BC. Under the constraints that the functions in the boundary conditions satisfy the Hölder condition, we reduce our problem to the singular integral equation with the Cauchy kernel. On the basis of this fact we study the existence and uniqueness theorem for the considered problem in the weighted space of Hölder continuous functions and construct its solution in a closed form.

In Section 2 we put the problem and give some properties of operators of classical and generalized fractional integration and differentiation. Section 3 is devoted to the reduction of the problem to a singular integral equation with the Cauchy kernel. In Section 4 we give a solution of this integral equation, and on the basis of this we prove the existence and uniquenes theorem for the solution of the problem in a weighted space of Hölder continuous functions and construct its explicit solution.

2. Problem and Operator of Fractional Integrals and Derivatives

To put the problem we present several notations. For $\alpha, \beta, \eta \in \mathbb{R}$ ($\alpha \neq 0$) generalized fractional integrals and derivatives $I_{0+}^{\alpha,\beta,\eta}\varphi$ and $I_{1-}^{\alpha,\beta,\eta}\varphi$ with the Gauss hypergeometric function $F(a, b; c; z)$ are defined for $x \in (0, 1)$ by the following ([9], [14, §18.2, 18.6]):

$$\left(I_{0+}^{\alpha,\beta,\eta}\varphi\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right)\varphi(t) \quad (\alpha > 0), \tag{3}$$

$$\left(I_{0+}^{\alpha,\beta,\eta}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+}^{\alpha+n,\beta-n,\eta-n}\varphi\right)(x) \quad (\alpha < 0,\ n = [-\alpha]+1), \tag{4}$$

$$\left(I_{1-}^{\alpha,\beta,\eta}\varphi\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_x^1 (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; \frac{t-x}{1-x}\right)\varphi(t) \quad (\alpha > 0), \tag{5}$$

$$\left(I_{1-}^{\alpha,\beta,\eta}\varphi\right)(x) = \left(-\frac{d}{dx}\right)^n \left(I_{1-}^{\alpha+n,\beta-n,\eta-n}\varphi\right)(x) \quad (\alpha < 0,\ n = [-\alpha]+1). \tag{6}$$

When $\beta = -\alpha$, (3)–(6) coincide to the following Riemann–Liouville fractional integrals and derivatives [14]:

$$\left(I_{0+}^{\alpha}\varphi\right)(x) = \left(I_{0+}^{\alpha,-\alpha,\eta}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\varphi(t) \quad (0 < x < 1;\ \alpha > 0), \tag{7}$$

$$\left(D_{0+}^{\alpha}\varphi\right)(x) = \left(I_{0+}^{-\alpha,\alpha,\eta}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\alpha}\varphi\right)(x) \quad (0 < x < 1;\ \alpha > 0,\ n = [\alpha]+1) \tag{8}$$

$$\left(I_{1-}^{\alpha}\varphi\right)(x) = \left(I_{1-}^{\alpha,-\alpha,\eta}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 (x-t)^{\alpha-1}\varphi(t) \quad (0 < x < 1;\ \alpha > 0), \tag{9}$$

$$\left(D_{1-}^{\alpha}\varphi\right)(x) = \left(I_{1-}^{-\alpha,\alpha,\eta}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{1-}^{n-\alpha}\varphi\right)(x) \quad (0 < x < 1;\ \alpha > 0,\ n = [\alpha]+1). \tag{10}$$

We denote the class of functions satisfying the Hölder condition of order λ on the interval $[0, 1]$ by $H^\lambda[0, 1]$ ($0 < \lambda < 1$) and its subclass by

$$H_0^\lambda[0,1] = \left\{\varphi(x) \in H^\lambda[0,1] :\ \varphi(0) = \varphi(1) = 0\right\}. \tag{11}$$

For a non-negative function $\rho(x)$ given on $[0, 1]$, $H_0^\lambda(\rho; [0, 1])$ denotes the class of Hölder functions $\varphi(x)$ such that $\rho(x)\varphi(x) \in H_0^\lambda[0, 1]$.

For the equation (1) we formulate the following problem:

Problem. Find a function $U(x, y) \in C(\overline{D}) \bigcup C^2(D)$, satisfying the equation (1) in the domain D and boundary conditions

$$U(x,0) = \tau(x) \quad (x \in [0,1]), \tag{12}$$

$$A\left(D_{0+}^{\alpha}U[\Theta_0(t)]\right)(x) + B\left(I_{1-}^{-\alpha-1/2,\beta,\alpha-1/2}U[\Theta_1(t)]\right)(x) = g(x) \quad (x \in (0,1)), \tag{13}$$

where $0 < \alpha < 1/2$ and $\beta \leqq 0$. Here $\tau(x)$ and $g(x)$ are given functions such as

$$\tau(x) \in H_0^{\lambda_1}(\rho; [0,1]) \cap C^2(0,1), \quad \left(\rho(x) = x^{-\alpha+1/2}, \quad \frac{1}{2} < \lambda_1 < 1\right), \tag{14}$$

$$g(x) \in H^{\lambda_2}[0,1] \cap C^2(0,1), \quad \left(-\alpha + \frac{1}{2} < \lambda_2 \leqq 1\right), \tag{15}$$

and A, B are given constants.

We find the solution of the problem (1), (12), (13) in the class of such functions that

$$\lim_{y \to 0-0} U_y(x,y) = \nu(x) \in H_0^{\lambda}(\rho; [0,1]), \tag{16}$$

where $\rho(x) = x^{-\alpha+1/2}$ and $1/2 - \alpha < \lambda < 1/2$.

For the solution of the formulated problem, we need the following properties of operators of fractional integrations and differentiations:

$$D_{0+}^{\alpha} I_{0+}^{\beta} f = I_{0+}^{\beta-\alpha} f \quad (0 < \alpha < \beta), \tag{17}$$

$$D_{0+}^{\alpha} D_{0+}^{\beta} f = D_{0+}^{\alpha+\beta} f \quad (\alpha > 0, \beta > 0), \tag{18}$$

$$\left(I_{0+}^{\alpha}\right)^{-1} f = I_{0+}^{-\alpha} f \equiv D_{0+}^{\alpha} f \quad (\alpha > 0), \tag{19}$$

$$I_{1-}^{\alpha,\beta,\eta} I_{1-}^{\gamma,\delta,\alpha+\eta} f = I_{1-}^{\alpha+\gamma,\beta+\delta,\eta} f \quad (\alpha, \gamma, \beta, \eta, \delta \in \mathbf{R}) \tag{20}$$

In the following we need the several results in [13] and [14].

Lemma 1. *Let $0 < \alpha < 1$, $0 < \lambda < 1$, $\lambda + \alpha < 1$ and $\rho(x) = x^{\mu}$, where $0 \leqq \mu < \lambda + 1$. If $\varphi(x) \in H_0^{\lambda}(\rho; [0,1])$, then $(I_{0+}^{\alpha}\varphi)(x) \in H_0^{\lambda+\alpha}(\rho; [0,1])$.*

Lemma 2. *Let $0 < \alpha < \lambda < 1$, $\lambda - \alpha < 1$ and $\rho(x) = x^{\mu}$, where $0 \leqq \mu < \lambda - \alpha + 1$. If $\varphi(x) \in H_0^{\lambda}(\rho; [0,1])$, then $(D_{0+}^{\alpha}\varphi)(x) \in H_0^{\lambda-\alpha}(\rho; [0,1])$.*

Lemma 3. *Let $0 < -\alpha < \lambda \leqq 1$ and $\eta > \beta - 1$. If $\varphi(x) \in H^{\lambda}[0,1]$, then $x^{\beta}(I_{0+}^{\alpha,\beta,\eta}\varphi)(x)$, $(1-x)^{\beta}(I_{1-}^{\alpha,\beta,\eta}\varphi)(x) \in H^{\lambda+\alpha}[0,1]$.*

Lemma 4. *Let $0 < -\alpha < \lambda \leqq 1$ and $\beta < \min[0, \eta+1]$. If $\varphi(x) \in H^{\lambda}[0,1]$, then $(I_{0+}^{\alpha,\beta,\eta}\varphi)(x)$, $(I_{1-}^{\alpha,\beta,\eta}\varphi)(x) \in H^{\min[\lambda+\alpha,-\beta]}[0,1]$.*

3. Reduction to a Singular Integral Equation

It is well known [1] that the solution of the Cauchy problem

$$U(x,0) = \tau(x) \quad (x \in [0,1]), \quad \lim_{y \to 0-0} U_y(x,y) = \nu(x) \quad (x \in (0,1))$$

for the equation (1) in the domain D has the form

$$U(x,y) = \tau\left(x + \frac{y^2}{2}\right) + \frac{y}{2} \int_0^1 \nu\left(x + \frac{y^2}{2}[1-2s]\right) s^{-1/2} ds. \tag{21}$$

Using (21), we have

$$U[\Theta_0(t)] = \tau(t) - \frac{\sqrt{t}}{2}\int\limits_0^1 \nu[t(1-s)]s^{-1/2}ds = \tau(t) - \Gamma\left(\frac{3}{2}\right)\left(I_{0+}^{1/2}\nu\right)(t), \tag{22}$$

$$U[\Theta_1(t)] = \tau(1) - \frac{\sqrt{1-t}}{2}\int\limits_0^1 \nu[1-(1-t)s]s^{-1/2}ds = -\frac{1}{2}\left(I_{1-}^{1,-1/2,-1}\nu\right)(t). \tag{23}$$

By (17) and (20), we have

$$\left(D_{0+}^{\alpha}I_{0+}^{1/2}\nu\right)(x) = \left(I_{0+}^{-\alpha+1/2}\nu\right)(x) \quad \left(0<\alpha<\frac{1}{2}\right), \tag{24}$$

$$\left(I_{1-}^{-\alpha-1/2,\beta,\alpha-1/2}I_{1-}^{1,-1/2,-1}\nu\right)(x) = \left(I_{1-}^{-\alpha+1/2,\beta-1/2,\alpha-1/2}\nu\right)(x) \quad \left(0<\alpha<\frac{1}{2},\ \beta \leqq 0\right). \tag{25}$$

Formulas (24) and (25) are valid for $\nu(x) \in H_0^{\lambda}(\rho;[0,1])$ with $0<\lambda<1/2$ and $\rho(x) = x^{-\alpha+1/2}$, and the fractional integrals $\left(I_{0+}^{-\alpha+1/2}\nu\right)(x)$ and $\left(I_{1-}^{-\alpha+1/2,\beta-1/2,\alpha-1/2}\nu\right)(x)$ are Hölder continuous functions. Then, if $\nu(x) \in H_0^{\lambda}(\rho;[0,1])$, from Lemmas 1 and 2 for $0<\lambda<1/2$ and $0<\alpha<1/2$ we obtain

$$\left(I_{0+}^{1/2}\nu\right)(x) \in H_0^{\lambda+1/2}(\rho;[0,1]), \quad \left(D_{0+}^{\alpha}I_{0+}^{1/2}\nu\right)(x) \in H_0^{\lambda-\alpha+1/2}(\rho;[0,1])$$

Then

$$\left(I_{0+}^{-\alpha+1/2}\nu\right)(x) \in H_0^{\lambda-\alpha+1/2}(\rho;[0,1]) \tag{26}$$

and (24) is correct. Further introducing the representation

$$\nu_1(x) = \left(I_{1-}^{1,-1/2,-1}\nu\right)(x) = \int\limits_x^1 \frac{\nu(t)}{\sqrt{1-t}}dt \in H^1[0,1]$$

and taking the conditions $0<\alpha<1/2$ and $\beta \leqq 0$ into consideration, we obtain from Lemma 3 that

$$\left(I_{1-}^{-\alpha-1/2,\beta,\alpha-1/2}\nu_1\right)(x) = \left(I_{1-}^{-\alpha+1/2,\beta-1/2,\alpha-1/2}\nu\right)(x) \in H^{1/2-\alpha}[0,1] \tag{27}$$

for $\beta=0$. For $\beta<0$ Lemma 4 implies the estimate

$$\left(I_{1-}^{-\alpha+1/2,\beta-1/2,\alpha-1/2}\nu\right)(x) \in H^{\min(1/2-\alpha,-\beta)}[0,1], \tag{28}$$

which guarantees the validity of the equation (25).

Substituting (22) and (23) into the boundary condition (13) and taking (24) and (25) into account, we have

$$A\sqrt{\pi}\left(I_{0+}^{-\alpha+1/2}\nu\right)(x) + B\left(I_{1-}^{-\alpha+1/2,\beta-1/2,\alpha-1/2}\nu\right)(x) = 2A\left(D_{0+}^{\alpha}\tau\right)(x) - 2g(x). \tag{29}$$

If $\nu(x) \in H_0^{\lambda}(\rho;[0,1])$, $\alpha<1/2$ and $-\alpha+1/2<\lambda<1/2$, then from (16), (19) and Lemma 2 we obtain

$$\left(I_{0+}^{-\alpha+1/2}\nu\right)^{-1}(x) = \left(I_{0+}^{\alpha-1/2}\nu\right)(x) = \left(D_{0+}^{1/2-\alpha}\nu\right)(x) \in H_0^{\lambda+\alpha-1/2}(\rho;[0,1]). \tag{30}$$

If $\tau(x) \in H_0^{\lambda_1}(\rho;[0,1])$, $\alpha < 1/2$ and $1/2 < \lambda_1 < 1$, then from (14), (19), (18) and Lemma 2 we have

$$\left(I_{0+}^{-\alpha+1/2}\right)^{-1}\left(D_{0+}^{\alpha}\tau\right)(x) = \left(D_{0+}^{1/2-\alpha}\right)\left(D_{0+}^{\alpha}\tau\right)(x) = \left(D_{0+}^{1/2}\tau\right)(x) \in H_0^{\lambda_1-1/2}(\rho;[0,1]). \quad (31)$$

Applying the operator $D_{0+}^{-\alpha+1/2}$ in the both sides of (29) and cosidering (30) and (31), we arrive at

$$A\sqrt{\pi}\nu(x) + B\left(D_{0+}^{-\alpha+1/2} I_{1-}^{-\alpha+1/2,\beta-1/2,\alpha-1/2}\nu\right)(x) = 2A\left(D_{0+}^{1/2}\tau\right)(x) - 2\left(D_{0+}^{-\alpha+1/2}g\right)(x),$$

or, at

$$A\sqrt{\pi}\nu(x) + B\left(I_{0+}^{\alpha-1/2,-\alpha+1/2,1} I_{1-}^{-\alpha+1/2,\beta-1/2,\alpha-1/2}\nu\right)(x) = 2A\left(D_{0+}^{1/2}\tau\right)(x) - 2\left(D_{0+}^{-\alpha+1/2}g\right)(x). \quad (32)$$

for $x \in (0,1)$. In [15] it was obtained that for $0 < p < 1$ and $r, s \in \mathbb{R}$

$$\left(I_{0+}^{-p,p,r} I_{1-}^{p,s,-p}\nu\right)(x) = \frac{\cos(\pi p)\nu(x)}{(1-x)^{p+s}} + \frac{\sin(\pi p)}{\pi}\int_0^1 \left(\frac{t}{x}\right)^p \frac{\nu(t)}{(1-t)^{p+s}(t-x)}dt \quad (0 < x < 1). \quad (33)$$

Applying (33) with $p = -\alpha + 1/2$ and $s = \beta - 1/2$, the formula (32) is written in the explicit form of integral equation

$$\left[A\sqrt{\pi} + B\sin(\pi\alpha)(1-x)^{\alpha-\beta}\right]\nu(x) + \frac{B\cos(\pi\alpha)}{\pi}\int_0^1 \left(\frac{t}{x}\right)^{-\alpha+1/2} \frac{\nu(t)}{(1-t)^{\beta-\alpha}(t-x)}dt$$

$$= 2A\left(D_{0+}^{1/2}\tau\right)(x) - 2\left(D_{0+}^{-\alpha+1/2}g\right)(x) \quad (x \in (0,1)) \quad (34)$$

4. Solution of the Singular Integral Equation and the Boundary Value Problem

Denoting in (34)

$$\varphi(x) = x^{-\alpha+1/2}(1-x)^{\alpha-\beta}\nu(x), \quad (35)$$

$$a(x) = A\sqrt{\pi} + B\sin(\pi\alpha)(1-x)^{\alpha-\beta}, \quad b(x) = B\cos(\pi\alpha)(1-x)^{\alpha-\beta}, \quad (36)$$

and

$$f(x) = 2x^{-\alpha+1/2}(1-x)^{\alpha-\beta}\left[A\left(D_{0+}^{1/2}\tau\right)(x) - \left(D_{0+}^{-\alpha+1/2}g\right)(x)\right], \quad (37)$$

we have the characteristic singular integral equation with the Cauchy kernel

$$a(x)\varphi(x) + \frac{b(x)}{\pi}\int_0^1 \frac{\varphi(t)}{t-x}dt = f(x) \quad (x \in (0,1)). \quad (38)$$

Since $\alpha - \beta > 0$, then $a(x), b(x) \in H^{\alpha-\beta}[0,1]$ from (26). If also $g(x) \in H^{\lambda_2}[0,1]$ and $-\alpha + 1/2 < \lambda_2 \leqq 1$, then by virtue of (7) and Lemma 3 we have

$$x^{-\alpha+1/2}\left(D_{0+}^{-\alpha+1/2}g\right)(x) = x^{-\alpha+1/2}\left(I_{0+}^{\alpha-1/2,-\alpha+1/2,\eta}g\right)(x) \in H^{\lambda_2+\alpha-1/2}[0,1]. \quad (39)$$

Then by (31) and (39) the function $f(x)$ in (37) satisfies

$$f(x) \in H^{\gamma}[0,1] \quad \text{with} \quad \gamma = \min\left[\alpha-\beta, -\alpha+\frac{1}{2}, \lambda_1-\frac{1}{2}, \lambda_2+\alpha-\frac{1}{2}\right]. \quad (40)$$

Suppose that

$$d(x) \equiv \left[A\sqrt{\pi} + B\sin(\pi\alpha)(1-x)^{\alpha-\beta}\right]^2 + \left[B\cos(\pi\alpha)(1-x)^{\alpha-\beta}\right]^2 \neq 0 \quad (x \in [0,1]). \tag{41}$$

Taking (16) and (35) into account, we have the function $\varphi(x)$ in the form

$$\varphi(x) = (1-x)^{\alpha-\beta}\varphi^*(x), \quad \varphi^*(x) = x^{-\alpha+1/2}\nu(x) \in H_0^\lambda[0,1]. \tag{42}$$

Thus we can find the solution $\varphi(x)$ of the equation (38) in the weighted Hölder class $H_0^\lambda\left((1-x)^{\alpha-\beta};[0,1]\right)$ on $[0,1]$ (see [14, §30.1]) for which we make use of the following statement.

The coefficient $G(x)$ of the Riemann boudary value problem corresponding to the integral equation (38) is given by the form

$$G(x) = \frac{a(x)-ib(x)}{a(x)+ib(x)} = \frac{A\sqrt{\pi}+B[\sin(\pi\alpha)-i\cos(\alpha\pi)](1-x)^{\alpha-\beta}}{A\sqrt{\pi}+B[\sin(\pi\alpha)+i\cos(\pi\alpha)(1-x)^{\alpha-\beta}}$$

$$= \frac{A\sqrt{\pi}-iBe^{i\pi\alpha}(1-x)^{\alpha-\beta}}{A\sqrt{\pi}+iBe^{-i\pi\alpha}(1-x)^{\alpha-\beta}} = e^{i\Theta(x)}, \tag{43}$$

where $\Theta(x) = \arg[G(x)]$. Then

$$G(0) = \frac{A\sqrt{\pi}-iBe^{i\pi\alpha}}{A\sqrt{\pi}+iBe^{-i\pi\alpha}} = e^{i\Theta}. \tag{44}$$

We choose the value $\arg[G(x)]$ such as

$$0 \leqq \arg[G(0)] = \arg\left[\frac{A\sqrt{\pi}-iBe^{i\pi\alpha}}{A\sqrt{\pi}+iBe^{-i\pi\alpha}}\right] < 2\pi. \tag{45}$$

$\Theta \equiv \Theta(0) = \arg[G(0)]$ is taken as

$$\Theta = 2\tan^{-1}\left\{\frac{-B\cos(\alpha\pi)}{A\sqrt{\pi}+B\sin(\alpha\pi)}\right\} \quad \text{if} \quad B[A\sqrt{\pi}+B\sin(\alpha\pi)] < 0;$$

$$\Theta = 2\pi - 2\tan^{-1}\left\{\frac{-B\cos(\alpha\pi)}{A\sqrt{\pi}+B\sin(\alpha\pi)}\right\} \quad \text{if} \quad B[A\sqrt{\pi}+B\sin(\alpha\pi)] > 0;$$

$$\Theta = 0 \quad \text{if} \quad B = 0;$$

$$\Theta = \pi \quad \text{if} \quad A\sqrt{\pi}+B\sin(\alpha\pi) = 0.$$

Futher, we have $G(1) = 1, \Theta(1) = \arg[G(1)] = 0$ and, thus $\left[\frac{\Theta(1)}{2\pi}\right] = 0$. By (42) $n_0 = 0, n_1 = 0$ and the index κ of the equation (38) is equal to -1:

$$\kappa = \left[\frac{\Theta(1)}{2\pi}\right] + n_0 + n_1 - 1 = -1. \tag{46}$$

Moreover,

$$\mu_0 = 1 - n_0 - \frac{\Theta(0)}{2\pi} = 1 - \frac{\Theta}{2\pi}; \quad \mu_1 = \frac{\Theta(1)}{2\pi} - \left[\frac{\Theta(1)}{2\pi}\right] - n_1 = 0; \tag{47}$$

$$Z_0(x) = \exp\left[\frac{1}{2\pi}\int_0^1 \frac{\Theta(t)dt}{t-x} + \frac{\Theta\log x}{2\pi}\right]. \tag{48}$$

Then by virtue of [14, Theorem 30.2] we have the following statement.

Theorem 1. *Let* $0 < \alpha < 1/2$, $\beta \leqq 0$, *and the functions* $\tau(x)$ *and* $g(x)$ *satisfy the corresponding conditions* (14) *and* (15). *Suppose that* $a(x)$, $b(x)$ *and* $f(x)$ *are given in* (36)–(37), *and* A *and* B *satisfy the condition* (41). *For the solvability of the equation* (38) *in the space* $H_0^\lambda\left((1-x)^{\alpha-\beta};[0,1]\right)$ *it is necessary and sufficient that*

$$\int_0^1 \frac{f(t)}{Z_0(t)t^{1-\theta/2\pi}}dt = 0. \tag{49}$$

When this condition is satisfied, the equation (38) *has a unique solution given by the formula*

$$\varphi(x) = \frac{a(x)f(x)}{d(x)} - \frac{bZ_0(x)}{\pi d(x)}\int_0^1 \left(\frac{x}{t}\right)^{1-\Theta/2\pi} \frac{f(t)}{Z_0(t)(t-x)}dt. \tag{50}$$

From the results in Section 4 and Theorem 1 the solvability of the boundary value problem proposed in Section 2 is given by the following theorem.

Theorem 2. *Let* $0 < \alpha < 1/2$, $\beta \leqq 0$, *and the functions* $\tau(x)$ *and* $g(x)$ *satisfy the corresponding conditions* (14) *and* (15), *and let the conditions* (41) *and* (49) *be fulfilled. Then the boundary value problem for the hypergeometric equation* (1) *with the conditions* (12)–(13) *has a unique solution* $U(x,y)$ *in the class* (16), *and the solution given by the formula* (21), *where*

$$\nu(x) = x^{\alpha-1/2}(1-x)^{\beta-\alpha}\varphi(x). \tag{51}$$

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