Stochastic Processes in Quantum Mechanics and Contemporary Macroeconomics

Yoshihiro Yamazaki*

Introduction

Probabilities play a very important role in quantum mechanics. In Schrödinger equation, imaginary numbers are interpreted as existing probabilities of quantum. Even after publishing the equation, Schrödinger continued to investigate the mathematics of probability. This research combined with Kolmogoroff's theory of probability. The theory of probability process was finally completed by a Japanese, Kiyoshi Ito. After WWII Nelson started stochastic quantum mechanics using geometric Brownian motion. This theory could solve many paradoxes which are included in normal quantum mechanics.

In economics, the concept of geometric Brownian motion was introduced in the end of 1960s. Investment theory adopted the stochastic process first. At almost the same time, Black-Scholes equation appeared. Contemporary macroeconomics, however, did not use Brownian-type stochastic process because the process could not describe business cycles. It used AR process with a unit root.

This paper shows this historical process of probabilities in quantum mechanics

*Faculty of Economics, Fukuoka University, Fukuoka, Japan

and economics. Then we analyze each characteristic of using stochastic processes. We also conclude the difference between quantum mechanics and contemporary macroeconomics.

1. Probability in Quantum Mechanics

Needless to mention, Schrödinger was one of the most important builders of quantum mechanics. He proposed the wave equation which is one of the two expression of quantum theory together with Heisenberg's matrix mechanics. Schrödinger, however, had a doubt on the interpretation of his own wave function. He talked of "Schrödinger's cat" and pointed out a paradox raised by the interpretation.

Schrödinger was against Born's statistical interpretation of wave equation. He tried to build a theory of stochastic process for himself to make another expression for quantum mechanics. His trial appeared as Schrödinger (1931).

At the same time, Kolmogoroff also researched the theory of stochastic process independently. His paper was published in the same year as Schrödinger's and appeared as Kolmogoroff (1931). And Kolmogoroff (1933) set the base of modern theory of probability. Both papers did not refer to Schrödinger's works. Kolmogoroff (1936) and Kolmogoroff (1937), however, quoted Schrödinger (1931).

After their works, Kiyoshi Ito completed the theory of stochastic process in the 1940s. Nelson (1966) proposed another expression of quantum mechanics that Schrödinger dreamt before using Ito's mathematics. Nelson's theory is called stochastic quantum mechanics.

In the theory, the position of a quantum follows this stochastic differential equation.

$$\mathrm{dX} = \mathrm{bdt} + \sqrt{\frac{\hbar}{2m}}\mathrm{dW} \tag{1}$$

The first term is the motion from average forward velocity field and the second is quantum fluctuations of spreading coefficient[1]. W is Wiener process which is normal stochastic variable of geometric Brownian motion.

$$\mathrm{m}\frac{1}{2}[DD_{*}X + D_{*}DX] = -\nabla V \tag{2}$$

This is Newton-Nelson equation. D and D_{*} are average forward and backward differentials. Geometric Brownian motion is irreversible. The equation (2) included past and future symmetrically. Because of this formation, we must introduce average forward velocity field and backward field at the same time.

Here when we introduce the probability density function P that means a quantum exists in the small space, we can obtain these two Fokker-Plank equations.

$$\frac{\partial P}{\partial t} = -\nabla bP + \frac{\hbar}{2m}\nabla^2 P \tag{3}$$

$$\frac{\partial P}{\partial t} = -\nabla b_{*}P - \frac{\hbar}{2m}\nabla^2 P \tag{4}$$

The summation and the difference are as follows.

[1] Here ħ is Dirac's constant, which means Planck's constant divided by 2π.

$$\frac{\partial P}{\partial t} + \nabla \frac{1}{2}(b + b_*)P = 0 \tag{5}$$

$$\frac{1}{2}(b - b_*)P = \frac{\hbar}{2m}\nabla^2 P \tag{6}$$

We then substitute the following two relations (7) and (8) for Newton-Nelson equation and the equation (9) follows.

$$D_*\mathrm{DX} = D_* b \cong \frac{\partial b}{\partial t} + b_*\nabla b - \frac{\hbar}{2m}\nabla^2 b \tag{7}$$

$$\mathrm{D}D_*\mathrm{X} = \mathrm{D}b \cong \frac{\partial b_*}{\partial t} + b\nabla b_* - \frac{\hbar}{2m}\nabla^2 b_* \tag{8}$$

$$\mathrm{m}\left[\frac{\partial}{\partial t}\frac{1}{2}(b + b_*) + \frac{1}{2}(b\nabla b_* + b_*\nabla b) - \frac{\hbar}{2m}\nabla^2(b - b_*)\right] = -\nabla V \tag{9}$$

Stochastic quantum mechanics is consisted of equations (5), (6) and (9). To simplify the system, we introduce flow velocity field v and diffusion velocity field u.

$$\mathrm{v} = \frac{1}{2}(b + b_*) \tag{10}$$

$$\mathrm{u} = \frac{1}{2}(b - b_*) \tag{11}$$

When we use equations (10) and (11), equations (5), (6) and (9) turn into equations (12), (13) and (14).

$$\frac{\partial P}{\partial t} + \nabla P v = 0 \tag{12}$$

$$\mathrm{uP} = \frac{\hbar}{2m} \nabla P \tag{13}$$

$$\mathrm{m}\frac{\partial v}{\partial t} + mv\nabla v - \mathrm{mu}\nabla \mathrm{u} - \frac{\hbar}{2}\nabla^2 u = -\nabla V \tag{14}$$

From equation (13), we can obtain equation (15).

$$\mathrm{u} = \frac{\hbar}{2m} \nabla lnP \tag{15}$$

When we substitute equation (15) for equation (14), we can eliminate u.

$$\mathrm{m}\frac{\partial v}{\partial t} + mv\nabla v - \frac{\hbar^2}{2m}\nabla \frac{\nabla^2\sqrt{P}}{\sqrt{P}} = -\nabla V \tag{16}$$

Here we have reached two non-linear partial differential equations (12) and (16). Then we substitute equation (17) for equations (12) and (16). We can obtain partial differential equations (18) and (19).

$$v = \frac{\nabla \mathrm{S}}{m} \tag{17}$$

$$\frac{\partial P}{\partial t} + \nabla \left[P \frac{\nabla \mathrm{S}}{m} \right] = 0 \tag{18}$$

$$\nabla \left[\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla^2\sqrt{P}}{\sqrt{P}} \right] = 0 \tag{19}$$

S is one of real constants in a wave function (20).

$$\Psi = \mathrm{R} \cdot \exp\left[\frac{\mathrm{iS}}{\hbar}\right] \tag{20}$$

As variable conversion (17) has the degree of freedom $S \rightarrow S + C(t)$, equation (19) can turn into equation (21).

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{P}}{\sqrt{P}} = 0 \tag{21}$$

When we use the function (20) and a variable conversion (22), we can obtain Schrödinger equation (23) at last.

$$\mathrm{P} = R^2 \tag{22}$$

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \mathrm{V}\Psi \tag{23}$$

We can describe the movement of quantum using an equation of classical mechanics when we assume stochastic process.

2. Introduction of Stochastic Process into Economics

Several years after quantum mechanics, stochastic process was also adopted in economics. Lucas (1971) and Hartman (1972) described a firm's investment behavior using stochastic process[2]. Black & Scholes (1973) and Merton (1973) derived option prices using Black-Scholes equation. Those happened almost in the same time.

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[2] Lucas became the advocate of macro rational expectation soon after this.

Now K is capital stock and X is productivity shock. X follows the stochastic differential equation (24). Here w is standard Brownian motion.

$$\mathrm{dX(t)} = \mu\big(\mathrm{X(t)}\big)\mathrm{dt} + \sigma\big(\mathrm{X(t)}\big)\mathrm{dw} \tag{24}$$

Equation (25) represents the accumulation process of capital stock. Gross investment and depreciation rate are I and δ respectively.

$$\mathrm{dK(t)} = \big(\mathrm{I(t)} - \delta \mathrm{K(t)}\big)\mathrm{dt} \tag{25}$$

A firm maximizes operating profit π minus investment cost c. Then the firm value V is expressed by equation (26). Here E_s and ρ are conditional expectation at time s and discount rate respectively.

$$\mathrm{V}\big(\mathrm{K(s)}, \mathrm{X(s)}\big) = \max_I E_s \int_0^\infty \big[\pi\big(K(t+s), X(t+s)\big) - c(I(t+s), K(t+s))\big] e^{-\rho t} dt \tag{26}$$

When we apply the basic equation of dynamic planning, we obtain equation (27). The left side is demanded rate of return and the right side is maximized rate of return. The maximized rate consists of net profit and capital gain of firm value.

$$\rho \mathrm{V(K, X)} = \max_I \left[\pi(K, X) - c(I, X) + \frac{1}{dt} E_s dV\right] \tag{27}$$

Now we apply Ito's formula to equation (27). Then we obtain Hamilton-Jacobi equation (28)[3]. Here g is generation operator.

$$V_s + \max_I [\mathcal{g}(s)V + \pi(K,X) - c(I,K)] = \rho V(K,X) \tag{28}$$

Because $V_s = 0$, equation (29) follows.

$$\max\left[\pi(K,X) - c(I.K) + (I - \delta K)V_K + \mu(X)V_X + \frac{1}{2}\sigma(X)^2 V_{XX}\right] = \rho V(K,X) \tag{29}$$

Marginal value of capital q is equal to V_K, which is called Tobin's q. When we differentiate equation (29), we obtain equation (30).

$$\pi_K(K,X) - c_K(I^*,K) - \delta q + q_K(I^* - \delta K) + \mu(X)V_{XK} + \frac{1}{2}\sigma(X)^2 = \rho V_K \tag{30}$$

Because $q = V_K$ is a function of K and X, we can derive equation (31) using Ito's formula.

$$E_t[dq] = q_K(I^* - \delta K)dt + \mu(X)V_{XK}dt + \frac{1}{2}\sigma(X)^2 V_{XXK}dt \tag{31}$$

When we substitute equation (31) for equation (3), we obtain equation (32).

$$\frac{E_t[dq]}{dt} + \pi_K(K,X) - c_K(I^*,K) = (\rho + \delta)q \tag{32}$$

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[3] Wave function derived from Hamilton-Jacobi equation assembles Schrödinger equation in its shape. Because of this, the treatment in classic mechanics is supposed to be the nearest one to quantum mechanics.

In the left side, the first term is capital gain, the second marginal return of capital and the third marginal cost of investment. In short, the left side is expected rate of return. The right side is gross rate of return on the lending of capital. When we set q like equation (33), we can prove this to be the solution of equation (30).

$$\mathrm{q(t)} = E_t \int_0^\infty \left[\pi_K\big(K(t+s), X(t+s)\big) - c_K(I(t+s), K(t+s))\right] e^{-(\rho+\delta)s} ds \tag{33}$$

We have explained that economics introduced stochastic process to its base of the theory a little later than quantum mechanics.

3. Probability in Contemporary Macroeconomics

Stochastic disturbances were introduced into economics as elementary factor of the theories in the late 1960s as above. They were also used in macro rational expectation model and entered real business cycle model.

In RBC model, a household's maximization problem is expressed in equation (34). Here U, C, L, K, r, w and δ are utility, consumption, labor, capital, rental price, wage rate and depreciation rate respectively.

$$\max U_t = \sum_{i=0} \beta^i \left[\ln(C_{t+i}) - \mu L_{t+i}^{\gamma+1}\right] \tag{34}$$

$$\mathrm{s.t.}\, K_{t+1} + C_t = r_t K_t + w_t L_t + (1-\delta) K_t$$

Lagrangian can be expressed like equation (35). From equation (35), we can obtain equations (36), (37) and (38).

$$\Lambda = \sum_{i=t} \left[\ln(C_t) - \mu L_t^{\gamma+1} + \lambda_i \{ r_i K_i + w_i L_i + (1-\delta) K_i - K_{i+1} - C_i \} \right] \quad (35)$$

$$\frac{1}{C_t} - \lambda_t = 0 \quad (36)$$

$$w_t \lambda_t - (\gamma + 1)\gamma \mu L_t^{\gamma} = 0 \quad (37)$$

$$\beta(r_{t+1} - \delta + 1)\lambda_{t+1} - \lambda_t = 0 \quad (38)$$

When we substitute equation (36) for equations (37) and (38), we obtain equations (39) and (40). Equation (39) is substitution between labor and consumption and equation (40) is Euler equation.

$$\frac{w_t}{C_t} = (\gamma + 1)\mu L_t^{\gamma} \quad (39)$$

$$\frac{C_{t+1}}{C_t} = \beta(r_{t+1} - \delta + 1) \quad (40)$$

Productivity shock is essentially important in RBC model. Technology level A is, however, not follow geometric Brownian motion but AR process like equation (41) because RBC model must describe business cycles.

$$\ln(A_{t+1}) = \rho \ln(A_t) + e_{t+1} \quad (41)$$

Coefficient ρ normally takes value between 0 and 1. This case is called a stationary process and productivity shocks are attenuated as time passes.The case of ρ =1 is called nonstationary process and productivity shocks have permanent effects on the technology level.

Conclusion

What is the difference of using stochastic process between quantum mechanics and economics? Why does the difference emerge? We should finally make them clear.

In quantum mechanics, stochastic process means quantum's fluctuation. The probability there is supposed realistic and objective. In economics, however, stochastic disturbances were introduced as error terms especially in econometric field. They were thought to be subjective and informational. Such stochastic disturbance gradually became indispensable element of economic theory. In this process, stochastic disturbances have acquired realistic and objective nature as real shocks like in quantum mechanics.

References

Black, F. & M. Scholes (1973), 'The Pricing of Options and Corporate Liabilities,' *Journal of Political Economy* 81, 537-54.

Hartman, R. (1972), 'The Effects of Price and Cost Uncertainty on Investment,' *Journal of Economic Theory* 5, 258-66.

Ito, K. (1942), 'On Stochastic Processes : Infinitely divisible Laws of Probability,' *Japan Journ.Math*. 18, 261-301.

Kidland, F. & E. Prescott (1982), 'Time to Build and Aggregate Fluctuations,' *Econometrica* 50, 1345-71.

Kolmogoroff, A. (1931), 'Über die analytischen Methoden in Wahrscheinlichkeitsrechnung,' *Math. Ann*. 104, 414-58.

Kolmogoroff, A. (1933), 'Grundbegriffe der Wahrscheinlichkeitsrechnung,' *Ergbn. d. Math*. 2, Heft 3, Springer-Verlag.

Kolmogoroff, A. (1936), 'Zur Theorie der Markoffschen Ketten,' *Math. Ann*. 112, 155-60.

Kolmogoroff, A. (1937), 'Zur Umkehrbarkeit der statistischen Naturgesetze,' *Math. Ann*. 113, 766-72.

Lucas, R. E. & E. C. Prescott (1971), 'Investment under Uncertainty,' *Econometrica* 39, 659-81.

Merton, R. C. (1973), ‘The Theory of Option Pricing,’ *The Bell Journal of Economics and Management Science* 4, 141-83.

Nagasawa, M. (1961), ‘The Adjoint Processes of Diffusion with Reflecting Barrier,’ *Kodai Math. Sem. Rep*. 13, 235-48.

Nagasawa, M. (1964), ‘Time Reversal of Markov Processes,’ *Nagoya Mathematical Journal* 24, 177-204.

Nelson, E. (1966), ‘Derivation of the Schrödinger Equation from Newtonian Mechanics,’ *Physical Review* 150, 1079-85.

Schrödinger, E. (1931), ‘Über die Umkehrung der Naturgesetze,’ *Sitzungsberchte der preussischen Akad. der Wissenschaften Phsikalisch- Mathematische Klasse*, 144-53.

Schöedinger, E. (1932), ‘Sur la théorie relativiste de l’ ëlectron et l’inter- prëtation de la mëcanique,’ *Ann. Inst. H. Poincarë* 2, 269-310.

Yasue, K. (1981), ‘Quantum Mechanics and Stochastic Control Theory,’ *Journal of Math. Phys*. 22, 1010-20.